Vol. 15 (núm 1) pp. 141 - 149
Enero - Junio 2024
https://doi.org/10.19053/01217488.v15.n1.2024.15357

A note on zeros of orthogonal polynomials generated by canonical transformations

Una nota sobre ceros de polinomios ortogonales generados por transformaciones canónicas

German Cuida,[1*] Alejandro Molano[1**]

Abstract

In this work, the behavior of zeros of orthogonal polynomials associated with canonical spectral transformations of weight functions on $[0,\infty)$ is studied. Namely, by means of standard techniques, we obtain interlacing properties for zeros associated with some particular cases of rational and Christoffel transformations.

Keywords: orthogonal polynomials, canonical transformations, zeros.

Resumen

En este trabajo se estudia el comportamiento de ceros de polinomios ortogonales asociados a transformaciones espectrales canónicas de funciones de peso sobre $[0,\infty)$. A saber, mediante técnicas estándar, obtenemos propiedades de entrelazado para ceros asociados a algunos casos particulares de transformaciones racionales y de Christoffel.

Palabras Clave: polinomios Ortogonales, transformaciones canónicas, ceros.

Recepción: 01-Diciembre-2022
Aceptación: 06-Marzo-2023

[1*] Universidad Pedagógica y Tecnológica de Colombia, Duitama, Colombia, Escuela de Matemáticas y Estadística. Dirección electrónica: german.cuida@uptc.edu.co
[1**] Universidad Pedagógica y Tecnológica de Colombia, Duitama, Colombia, Escuela de Matemáticas y Estadística. Dirección electrónica: luis.molano01@uptc.edu.co

1. Introduction

Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of monic polynomials orthogonal with respect to a weight function ω en $(0,\infty)$. Consider the weight function $\rho\omega$ where

$$\rho(x)=\frac{\prod_{i=1}^{k}(x-\zeta_i)^{m_i}}{\prod_{j=1}^{p}(x-\eta_j)^{q_j}}, \tag{1}$$

where $k,p\geq 1,\ \zeta_i,\eta_j<0$ and $m_i,q_j\in\mathbb{Z}^+\cup\{0\}$. When $q_j=0$ for every j, the function ρ produces a perturbation on the weight ω, known in the literature as *Christoffel transformation*, introduced in [1], (see also [2], [3] and [4]). Orthogonal polynomials associated with this kind of perturbations have been widely studied, in particular analytic properties associated to zeros and asymptotic behavior, (see for instance [5], [6], [7], [8], [9], [10] and [11]). On the other hand, if $m_i=0$ for every i, ρ produces a very particular case of a *Geronimus transformation*, introduced in a general way in [12] and [13], and related with mechanical quadrature, numerical analysis and physics problems concerning non-isospectral discrete-time Volterra chains. See [14], [15], [16], [17], [18], [8], [19] and [20] for recent developments in analytic and asymptotic properties as well as electrostatic models. Finally, if in (1) there exist i,j such that $m_iq_j\neq 0$, we get a *rational transformation* of the weight. In [21] can be seen a deep study about asymptotic behavior of orthogonal polynomials with respect to this kind of perturbation when ω is the classical Laguerre weight. For a general treatment [16], [22], [23], [24], [10] and [11], are highly recommended.

Special cases of transformations outlined above are the focus of this paper. In particular, this contribution has to do with location of zeros of polynomials associated with this kind of perturbation in some very particular cases. In this way, the structure of this manuscript is as follows. In Section 2, we present some basic elements of the theory and auxiliary results. In Section 3 we discuss some algebraic connections and interlacing properties of zeros of polynomials associated to particular cases of Christoffel transformations when the weight is perturbed by linear factors $(x-\zeta_1)$ and $(x-\zeta_2)$, with $\zeta_1\neq\zeta_2$, and $\zeta_1,\zeta_2<0$. Finally, in section 4 we consider interlacing properties of zeros of polynomials orthogonal with respect to rational perturbations, namely, we consider perturbations with the rational functions $\dfrac{1}{x-\nu}$ and $\dfrac{(x-\zeta)}{(x-\nu)}$, with $\zeta\neq\nu$ and $\zeta,\nu<0$.

2. Preliminaries

Let $\mathbf{P}$ be the linear space of polynomials with complex coefficients. $\mathbf{P}_n$ will denote the linear subspace of polynomials of degree at most n. Let u be a linear functional in the algebraic dual space of $\mathbf{P}$. It will be denoted $\mathbf{P}'$. $\langle u,p\rangle$ is the action of the linear functional u on the polynomial $p\in\mathbf{P}$. For $u\in\mathbf{P}'$, the sequence $\{u_n\}_{n\geq 0}$, $u_n=\langle u,x^n\rangle$, is said to be the respective *moment sequence*. We define the *Hankel determinant of order* $n+1$ for $\Delta_n=|(u_{i+j})_{i,j=0}^{n}|$. Also, u is so called **quasi-definite** or **regular** if $\Delta_n\neq 0$ for $n\geq 0$, and it is called **positive-definite** if $\langle u,\pi(x)\rangle>0$ for every nonzero and non-negative real polynomial π.

Theorem 1 *([2]). u is positive definite if and only if their moments are real and $\Delta_n>0$ for $n\geq 0$.*

If u is positive-definite, then there exists a positive Borel measure μ supported on an infinite set $E\subseteq\mathbb{R}$ such that u has an integral representation

$$\langle u,p\rangle=\int_E p(x)d\mu(x),\quad p\in\mathbf{P}.$$

Given a quasi-definite linear functional u on the space $\mathbf{P}(\mathbb{R})$ of polynomials with real coefficient, a bilinear form $\langle,\rangle_u:\mathbf{P}(\mathbb{R})\times\mathbf{P}(\mathbb{R})\to\mathbb{R}$ is defined as $\langle p,q\rangle_u:=\langle u,pq\rangle$. If u is positive definite then the bilinear form is an inner product on $\mathbf{P}(\mathbb{R})$ and, it is usual,

$$\begin{aligned}\|p\|_u&=\langle p,p\rangle_u^{1/2}=\left\langle u,p^2\right\rangle^{1/2}\\&=\left(\int_E p^2(x)d\mu(x)\right)^{1/2},\end{aligned}$$

represents the induced norm.

Definition 1 *A sequence $\{P_n\}_{n\geq 0}$ is called an orthogonal polynomial sequence, (OPS in short), with respect to a moment functional u if for $n,m\geq 0$, i). P_n is a polynomial of degree n; ii). $\langle u,P_nP_m\rangle=0$, for $n\neq m$, and iii). $\left\langle u,P_n^2\right\rangle\neq 0$.*

If the leading coefficient of P_n is 1 for every $n\geq 0$, then $\{P_n\}_{n\geq 0}$ is said to be a *monic orthogonal polynomial sequence, (MOPS in short).*

proposition 1 *([2]). Let u be a moment functional. u is quasi-definite if and only if there exists an OPS $\{P_n\}_{n\geq 0}$ with respect to the functional*

Theorem 2 (Favard's theorem) *([2]). Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials. $\{P_n\}_{n\geq 0}$ is a MOPS with respect to a quasi-definite linear functional u if and only if there exist sequences of numbers $\{\beta_n\}_{n\geq 1}$ and $\{\gamma_n\}_{n\geq 1}$, with $\gamma_n\neq 0$ for $n\geq 1$, such that*

$$xP_n(x)=P_{n+1}(x)+\beta_nP_n(x)+\gamma_nP_{n-1}(x),\ \ n\geq 1, \tag{2}$$

$$P_0(x)=1,\ P_1(x)=x-\beta_0.$$

On the other hand,

$$\beta_n=\frac{\left\langle u,xP_n^2\right\rangle}{\left\langle u,P_n^2\right\rangle},\ n\geq 0,$$

$$\gamma_n=\frac{\left\langle u,xP_nP_{n-1}\right\rangle}{\left\langle u,P_{n-1}^2\right\rangle}=\frac{\left\langle u,P_n^2\right\rangle}{\left\langle u,P_{n-1}^2\right\rangle},\ n\geq 1.$$

If $\{x_{n,j}\}_{j=1}^{n}$ are the zeros of the polynomial P_n we enunciate the following result.

Theorem 3 *([2]). Let I be the support of a positive-definite linear functional u and $\{P_n\}_{n\geq 0}$ the respective MOPS. Then, i). The zeros of P_n are real, simple and located in the interior of the convex hull of I. ii). (Interlacing property). The zeros of P_n and P_{n+1} mutually separate each other, i.e. if $\{x_{n,j}\}_{j=1}^{n}$ are the n zeros of the polynomial P_n, arranged in an increasing order, then $x_{n+1,j} < x_{n,j} < x_{n+1,j+1}$, $1 \leq j \leq n$.*

We will consider the next tool, useful to deduce interlacing properties of zeros.

lemma 1 *(See [25]). Let $r_n(x) = (x-x_1)\cdots(x-x_n)$ and $r_{n-1}(x) = (x-y_1)\cdots(x-y_{n-1})$ be polynomials with real and interlacing zeros*

$$x_1 < y_1 < x_2 < \cdots < x_{n-1} < y_{n-1} < x_n,$$

Then for any real constant C the polynomial $R_n(x) = r_n(x) + Cr_{n-1}(x)$ has n real zeros $\xi_1 < \xi_2 < \cdots < \xi_n$ which interlace with both the zeros of $r_n(x)$ and $r_{n-1}(x)$ in the next way: if $C > 0$,

$$\xi_1 < x_1 < y_1 < \xi_2 < x_2 < y_2 < \cdots < \xi_{n-1} < x_{n-1} < y_{n-1} < \xi_n < x_n,$$

but if $C < 0$

$$x_1 < \xi_1 < y_1 < x_2 < \xi_2 < \cdots < x_{n-1} < \xi_{n-1} < y_{n-1} < x_n < \xi_n,$$

3. Christoffel Transformations

If u is a positive-definite linear functional, the respective Borel measure $d\mu$ is supported on $[a,b]$ and if $\zeta_i \notin (a,b)$, $i = 1,2,\ldots,k$, the measure $d\mu^* = \prod_{i=1}^{k}|x-\zeta_i|\,d\mu$ is called a canonical *Christoffel Transformation*. Concerning to the relation between polynomials orthogonal with respect to μ and μ_* in the particular case $k = 1$, we get the next result.

proposition 2 *([2]). Let $\{P_n\}$ be the MOPS with respect to μ, supported in the interval $[a,b]$. If $\zeta \leq a$, is not a zero of P_n, for every $n \geq 1$, then $\left\{P_n^{[1]}\right\}$, the monic sequence orthogonal with respect to $d\mu^* = (x-\zeta)d\mu$, satisfies*

$$P_n^{[1]}(x) = (x-\zeta)^{-1}\left(P_{n+1}(x) - \frac{P_{n+1}(\zeta)}{P_n(\zeta)}P_n(x)\right).$$

In addition, if $\left\{x_{n,i}^{[1]}\right\}_{i=1}^{n}$ are the real and simple zeros of $P_n^{[1]}$ then

$$x_{n,i} < x_{n,i}^{[1]}, \quad i = 1,\ldots,n. \tag{3}$$

In the sequel, let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of monic polynomials orthogonal with respect to the inner product

$$\langle r,q\rangle_\omega = \int_0^\infty r(x)q(x)d\mu_\omega, \tag{4}$$

where $d\mu_\omega = \omega(x)dx$, and ω is a weight function on $(0,\infty)$. As before, $\{x_{n,i}\}_{i=1}^{n}$ represents the zeros of P_n. In this way, let $\left\{P_n^{[k]}\right\}$ be the sequence of monic polynomials orthogonal with respect to

$$\langle p,q\rangle_k = \int_0^\infty p(x)q(x)(x-\zeta)^k w(x)dx = \int_0^\infty p(x)q(x)d\mu_{\omega,k}, \tag{5}$$

$\zeta < 0$, $k \geq 0$. For every n, $P_n^{[0]} := P_n$ and $\mu_{\omega,0} := \mu_\omega$. Also, $\|.\|_k$ represents the induced norm for (5), with $\|.\|_0 := \|.\|$, the latter, the norm induced by (4).

proposition 3 *(see [7]). For $k \geq 1$*

$$(x-\zeta)P_n^{[k]}(x) = P_{n+1}^{[k-1]}(x) - \frac{P_{n+1}^{[k-1]}(\zeta)}{P_n^{[k-1]}(\zeta)}P_n^{[k-1]}(x), \tag{6}$$

moreover

$$\left\|P_n^{[k]}\right\|_k^2 = (-1)^k\prod_{j=1}^{k}\left(\frac{P_{n+1}^{[k-j]}(\zeta)}{P_n^{[k-j]}(\zeta)}\right)\|P_n\|^2. \tag{7}$$

proposition 4 *Let $\left\{x_{n,i}^{[k]}\right\}_{i=1}^{n}$ the zeros of $P_n^{[k]}$ arranged in an increasing order. It holds that*

$$x_{n,i}^{[p]} < x_{n,i}^{[q]}, \quad i = 1,\ldots,n, \tag{8}$$

with $p,q \in \mathbb{N}\cup\{0\}$ and $p < q$.

proof 1 *Notice that from (3), we know how zeros of members of the MOPS associated with a weight on $[0,\infty)$, are interlaced with the zeros of members of the MOPS associated with the weight perturbed by $(x-\zeta)$. As a consequence we get*

$$x_{n,i}^{[k-1]} < x_{n,i}^{[k]},$$

for $k \geq 0$.

Next, we are going to obtain explicitly the three terms recurrence relation, (TTRR in short), that the sequence $\left\{P_n^{[k]}\right\}$ satisfies. To do that, we expand $xP_{n-1}^{[k]}$ in terms of $\left\{P_n^{[k]}\right\}$, namely

$$xP_{n-1}^{[k]}(x) = \sum_{i=0}^{n} a_{ni}P_i^{[k]}(x),$$

$$a_{n,i} = \frac{\int_0^\infty xP_{n-1}^{[k]}(x)P_i^{[k]}(x)(x-\zeta)^k w(x)dx}{\int_0^\infty (P_i^{[k]}(x))^2(x-\zeta)^k w(x)dx}.$$

It is clear that $a_{nk} = 0$ for $0 \leq k < n-3$. Then

$$xP_{n-1}^{[k]}(x) = P_n^{[k]}(x) + c_n^{[k]}P_{n-1}^{[k]}(x) + \lambda_n^{[k]}P_{n-2}^{[k]}(x).$$

According to explicit formulas for coefficients in (2) we get

$$\lambda_n^{[k]} = \frac{\left\|P_n^{[k]}\right\|_k^2}{\left\|P_{n-1}^{[k]}\right\|_k^2},$$

and

$$c_n^{[k]} = \frac{\int_0^\infty x(P_{n-1}^{[k]}(x))^2(x-\zeta)^k w(x)dx}{\left\|P_{n-1}^{[k]}\right\|_k^2}.$$

By means of (7) we get

$$\lambda_n^{[k]} = \frac{\|P_{n-1}\|^2}{\|P_{n-2}\|^2}\prod_{j=1}^{k}\frac{P_n^{[k-j]}(\zeta)P_{n-2}^{[k-j]}(\zeta)}{\left(P_{n-1}^{[k-j]}(\zeta)\right)^2}, \tag{9}$$

and

$$c_n^{[k]} = \frac{\int_0^\infty x(P_{n-1}^{[k]}(x))^2(x-\zeta)^k w(x)dx}{(-1)^k\prod_{j=1}^{k}\left(\frac{P_n^{[k-j]}(\zeta)}{P_{n-1}^{[k-j]}(\zeta)}\right)\|P_{n-1}\|^2}. \tag{10}$$

In particular, for $k=1$ we obtain

$$\lambda_n^{[1]} = \frac{\left\|P_n^{[1]}\right\|_1^2}{\left\|P_{n-1}^{[1]}\right\|_1^2},$$

and

$$c_n^{[1]} = \frac{\int_0^\infty x(P_{n-1}^{[1]}(x))^2(x-\zeta)w(x)dx}{\left\|P_{n-1}^{[1]}\right\|_1^2},$$

moreover

$$\begin{aligned} c_n^{[1]} &= \frac{\int_0^\infty (P_{n-1}^{[1]}(x))^2(x-\zeta)^2 w(x)dx}{\left\|P_{n-1}^{[1]}\right\|_1^2} + \zeta \\ &= \frac{\int_0^\infty \left(P_n(x) - \frac{P_n(\zeta)}{P_{n-1}(\zeta)}P_{n-1}(x)\right)^2 w(x)dx}{\left\|P_{n-1}^{[1]}\right\|_1^2} + \zeta \\ &= \frac{\|P_n\|^2}{\left\|P_{n-1}^{[1]}\right\|_1^2} + \frac{\left(\frac{P_n(\zeta)}{P_{n-1}(\zeta)}\right)^2\|P_{n-1}\|^2}{\left\|P_{n-1}^{[1]}\right\|_1^2} + \zeta. \end{aligned}$$

Here we have used (6). From (7), we finally obtain

$$c_n^{[1]} = -\frac{P_{n-1}(\zeta)\|P_n\|^2}{P_n(\zeta)\|P_{n-1}\|^2} - \frac{P_n(\zeta)}{P_{n-1}(\zeta)} + \zeta. \tag{11}$$

and

$$\begin{aligned} \lambda_n^{[1]} &= \frac{\frac{P_{n+1}(\zeta)}{P_n(\zeta)}\|P_n\|^2}{\frac{P_n(\zeta)}{P_{n-1}(\zeta)}\|P_{n-1}\|^2} \\ &= \frac{P_{n-1}(\zeta)P_{n+1}(\zeta)\|P_n\|^2}{(P_n(\zeta))^2\|P_{n-1}\|^2}. \end{aligned} \tag{12}$$

Summarizing, we get the following result.

proposition 5 *The sequence* $\left\{P_n^{[k]}\right\}$ *satisfies the TTRR*

$$P_n^{[k]}(x) = (x-c_n^{[k]})P_{n-1}^{[k]} - \lambda_n^{[k]}P_{n-2}^{[k]}(x), \tag{13}$$

with $c_n^{[k]}$ *and* $\lambda_n^{[k]}$ *defined in (9) and (10). In particular, when* $k=1$, *we get*

$$c_n^{[1]} = -\frac{P_{n-1}(\zeta)\|P_n\|^2}{P_n(\zeta)\|P_{n-1}\|^2} - \frac{P_n(\zeta)}{P_{n-1}(\zeta)} + \zeta,$$

and

$$\lambda_n^{[1]} = \frac{P_{n-1}(\zeta)P_{n+1}(\zeta)\|P_n\|^2}{(P_n(\zeta))^2\|P_{n-1}\|^2}.$$

Now, let $\{Q_n\}$ be the monic sequence of polynomials orthogonal with respect to

$$\langle p,q\rangle = \int_0^\infty p(x)q(x)(x-\zeta_1)(x-\zeta_2)d\mu_\omega,$$

$\zeta_1 < \zeta_2 < 0$, and let $\left\{P_n^{[1,1]}\right\}$, $\left\{P_n^{[1,2]}\right\}$ be the MOPS associated with the weights $(x-\zeta_1)\omega$ and $(x-\zeta_2)\omega$ respectively. Also, $\|.\|_{[1,1]}$ and $\|.\|_{[1,2]}$ will denote the respective induced norms. By using of (6) with $k=1$, we get

$$(x-\zeta_2)P_n^{[1,2]}(x) = P_{n+1}(x) - \frac{P_{n+1}(\zeta_2)}{P_n(\zeta_2)}P_n(x), \tag{14}$$

$$(x-\zeta_1)P_n^{[1,1]}(x) = P_{n+1}(x) - \frac{P_{n+1}(\zeta_1)}{P_n(\zeta_1)}P_n(x), \tag{15}$$

and

$$(x-\zeta_1)Q_n(x) = P_{n+1}^{[1,2]}(x) - \frac{P_{n+1}^{[1,2]}(\zeta_1)}{P_n^{[1,2]}(\zeta_1)}P_n^{[1,2]}(x). \tag{16}$$

Multiplying on both sides of (16) by $(x-\zeta_2)$ we get

$$\begin{aligned} &(x-\zeta_2)(x-\zeta_1)Q_n(x) \\ &= (x-\zeta_2)P_{n+1}^{[1,2]}(x) - \frac{P_{n+1}^{[1,2]}(\zeta_1)}{P_n^{[1,2]}(\zeta_1)}(x-\zeta_2)P_n^{[1,2]}(x) \\ &= P_{n+1}(x) + \left(-\frac{P_{n+1}(\zeta_2)}{P_n(\zeta_2)} - \frac{P_{n+1}^{[1,2]}(\zeta_1)}{P_n^{[1,2]}(\zeta_1)}\right)P_{n+1}(x) \\ &\quad + \left(\frac{P_{n+1}(\zeta_2)}{P_n(\zeta_2)}\right)\left(\frac{P_{n+1}^{[1,2]}(\zeta_1)}{P_n^{[1,2]}(\zeta_1)}\right)P_n(x), \\ &= P_{n+1}(x) + \gamma_n(\zeta_1,\zeta_2)P_{n+1}(x) + \rho_n(\zeta_1,\zeta_2)P_n(x), \end{aligned}$$

then we get the following result.

lemma 2 *For every n*

$$\begin{aligned} &(x-\zeta_2)(x-\zeta_1)Q_n(x) \\ &= P_{n+1}(x) + \gamma_n(\zeta_1,\zeta_2)P_{n+1}(x) + \rho_n(\zeta_1,\zeta_2)P_n(x), \end{aligned} \tag{17}$$

with

$$\gamma_n(\zeta_1,\zeta_2) = -\frac{P_{n+2}(\zeta_2)}{P_n(\zeta_2)} - \frac{P_{n+1}^{[1,2]}(\zeta_1)}{P_n^{[1,2]}(\zeta_1)},$$

and

$$\rho_n(\zeta_1,\zeta_2) = \frac{P_{n+1}(\zeta_2)}{P_n(\zeta_2)} \frac{P_{n+1}^{[1,2]}(\zeta_1)}{P_n^{[1,2]}(\zeta_1)}.$$

Notice that $\gamma_n(\zeta_1,\zeta_2), \rho_n(\zeta_1,\zeta_2) > 0$. Connection formula (17) is relevant since allow us to build every polynomial Q_n in terms of known data, that is, in terms of members of the original MOPS $\{P_n\}$.

On the other hand, if we expand $(x-\zeta_1)P_n^{[1,1]}$ by means of the basis $\left\{P_n^{[1,2]}\right\}$, we obtain

$$(x-\zeta_1)P_n^{[1,1]} = \sum_{i=0}^{n+1} a_{ni} P_i^{[1,2]}(x),$$

with

$$a_{ni} = \frac{\int_0^\infty (x-\zeta_1)P_n^{[1,1]}(x)P_i^{[1,2]}(x-\zeta_2)d\mu_\omega}{\left\|P_i^{[1,2]}\right\|^2_{[1,2]}}.$$

By orthogonality with respect to $(x-\zeta_1)\omega$, $a_{ni}=0$ for $i = 0,\ldots,n-2$. Formulas (14) and (15) allows us to obtain

$$\begin{aligned} a_{nn} &= -\frac{\|P_{n+1}\|^2 + (\frac{P_{n+1}(\zeta_1)}{P_n(\zeta_1)})(\frac{P_{n+1}(\zeta_2)}{P_n(\zeta_2)})\|P_n\|^2}{\frac{P_{n+1}(\zeta_2)}{P_n(\zeta_2)}\|P_n\|^2} \\ &= -\frac{P_n(\zeta_2)}{P_{n+1}(\zeta_2)}\frac{\|P_{n+1}\|^2}{\|P_n\|^2} - \frac{P_{n+1}(\zeta_1)}{P_n(\zeta_1)} \end{aligned}$$

and

$$\begin{aligned} a_{n,n-1} &= \frac{\frac{P_{n+1}(\zeta_1)}{P_n(\zeta_1)}\|P_n\|^2}{\frac{P_n(\zeta_2)}{P_{n-1}(\zeta_2)}\|P_{n-1}\|^2} \\ &= \frac{P_{n+1}(\zeta_1)P_{n-1}(\zeta_2)}{P_n(\zeta_1)P_n(\zeta_2)}\frac{\|P_n\|^2}{\|P_{n-1}\|^2}. \end{aligned}$$

Then we get the formula

$$\begin{aligned} &(x-\zeta_1)P_n^{[1,1]}(x) \\ &= P_{n+1}^{[1,2]}(x) \\ &- \left(\frac{P_n(\zeta_2)}{P_{n+1}(\zeta_2)}\frac{\|P_{n+1}\|^2}{\|P_n\|^2} + \frac{P_{n+1}(\zeta_1)}{P_n(\zeta_1)}\right)P_n^{[1,2]}(x) \\ &- \frac{P_{n+1}(\zeta_1)P_n(\zeta_2)}{P_n(\zeta_1)P_{n+1}(\zeta_2)}\frac{\|P_n\|^2}{\|P_{n-1}\|^2}P_{n-1}^{[1,2]}(x). \end{aligned}$$

Now, by using the TTRR (13) for the sequence $\left\{P_n^{[1,2]}\right\}$ we get the following result.

proposition 6 *For every n,*

$$(x-\zeta_1)P_n^{[1,1]}(x) = (x-\gamma_n^{[1,2]})P_n^{[1,2]}(x) + \eta_n^{[1,2]}P_{n-1}^{[1,2]}(x), \tag{18}$$

where

$$\gamma_n^{[1,2]} = \frac{P_{n+1}(\zeta_1)}{P_n(\zeta_1)} - \frac{P_{n+1}(\zeta_2)}{P_n(\zeta_2)} + \zeta_2, \tag{19}$$

and

$$\begin{aligned} &\eta_n^{[1,2]} \\ &= \frac{\|P_n\|^2}{\|P_{n-1}\|^2}\frac{P_{n-1}(\zeta_2)}{P_n(\zeta_2)}\left[\frac{P_{n+1}(\zeta_1)}{P_{n+1}(\zeta_1)} - \frac{P_{n+1}(\zeta_2)}{P_n(\zeta_2)}\right]. \end{aligned} \tag{20}$$

To find out the sign of each coefficient in the above formulas, we present the following useful result.

lemma 3 *For $n \in \mathbb{N}$, $R_n(x) = \dfrac{P_{n+1}(x)}{P_n(x)}$ is a increasing function on $(-\infty,0]$.*

proof 2 *It is enough to prove that the first derivative of $R_n(x)$ is positive on $(-\infty,0)$. Indeed*

$$R_n'(x) = \frac{p_{n+1}'(x)p_n(x) - p_{n+1}(x)p_n'(x)}{(p_n(x))^2},$$

In this way, and as a consequence, we can deduce the next important information.

corollary 1 *For every n, $\zeta_1 < \zeta_2 < 0$, and $\gamma_n^{[1,2]}$, $\eta_n^{[1,2]}$ defined in (19) and (20) respectively, it holds $\gamma_n^{[1,2]} < 0$ and $\eta_n^{[1,2]} > 0$.*

Let $\left\{x_{n,i}^{[1,1]}\right\}_{i=1}^n$ and $\left\{x_{n,i}^{[1,2]}\right\}_{i=1}^n$ be the zeros of $P_n^{[1,1]}$ and $P_n^{[1,2]}$ respectively, and all arranged in an increasing order. The connection formula (18) can be written as

$$\begin{aligned} &(x-\zeta_1)P_n^{[1,1]}(x) \\ &= xP_n^{[1,2]}(x) - \gamma_n^{[1,2]}\left(P_n^{[1,2]}(x) - \frac{\eta_n^{[1,2]}}{\gamma_n^{[1,2]}}P_{n-1}^{[1,2]}(x)\right), \end{aligned} \tag{21}$$

and we consider the sequence of monic polynomials $\{D_n\}$ defined as follows:

$$D_n(x) = P_n^{[1,2]}(x) - \frac{\eta_n^{[1,2]}}{\gamma_n^{[1,2]}}P_{n-1}^{[1,2]}(x). \tag{22}$$

This family is *quasi-orthogonal* with respect to the weight $(x-\zeta_2)\omega$ on $[0,\infty)$ in the next sense:

Definition 2 *Let R_n be a polynomial of exact degree n. If ω is a weight function on interval $[a,b]$, and R_n satisfies the conditions*

$$\int_a^b x^k R_n(x)\omega(x)dx = 0,$$

$k = 0,\ldots,n-r-1$, then R_n is quasi-orthogonal of order r with respect to ω on $[a,b]$.

Next, some consequences of this definition.

Theorem 4 *(See [26]). If $\{P_n\}$ is a OPS with respect to ω on $[a,b]$, R_n is quasi-orthogonal of order r with respect to ω on $[a,b]$ if there exist numbers $c_{n,i}$, $i=1,\dots,r$, $c_{n,r}\neq 0$, such that*

$$R_n(x) = P_n(x) + \sum_{k=1}^{r} c_{n,k} P_{n-k}(x).$$

Theorem 5 *(See [27]). If R_n is quasi-orthogonal of order r with respect to ω on $[a,b]$, the zeros are real, simple and at least $n-r$ lie in (a,b).*

Let $\{\varphi_{n,i}\}_{i=1}^{n}$ be the zeros of D_n, arranged in an increasing order. By Lemma 1 since $x_{n,i}^{[1,2]} < x_{n-1,i}^{[1,2]} < x_{n,i+1}^{[1,2]}$, $i=1,\dots,n-1$, from (22), we get the following result.

lemma 4 *For every $n\geq 1$, $\varphi_{n,1} < x_{n,1}^{[1,2]}$, and*

$$x_{n-1,i}^{[1,2]} < \varphi_{n,i+2} < x_{n,i+1}^{[1,2]}$$

for $i=1,\dots,n-1$.

$$\varphi_{n,1} < x_{n,1}^{[1,2]} < x_{n-1,1}^{[1,2]} < \varphi_{n,2} < x_{n,2}^{[1,2]} < \cdots < \varphi_{n,n-1} < x_{n,n-1}^{[1,2]} < x_{n-1,n-1}^{[1,2]} < \varphi_{n,n} < x_{n,n}^{[1,2]}.$$

If $0 < \varphi_{n,1} < x_{n,1}^{[1,2]}$, and taking into account Lemma 4, we can use formula (21) and Lemma 1 to prove the following interlacing properties.

proposition 7 *If $0<\varphi_{n,1}$, then*

$$x_{n,i}^{[1,1]} < x_{n,i}^{[1,2]},$$

for $i=1,\dots,n$.

On the other hand, if $\varphi_{n,1}<0$, notice that if we use (21) for $x=0$ and $x=x_{n,1}^{[1,2]}$ we get

$$P_n^{[1,1]}(0) P_n^{[1,1]}\left(x_{n,1}^{[1,2]}\right) = -\frac{\left(\gamma_n^{[1,2]}\right)^2}{\zeta_1\left(x_{n,1}^{[1,2]}-\zeta_1\right)} D_n(0) D_n\left(x_{n,1}^{[1,2]}\right),$$

and from Lemma 4 we know that $\varphi_{n,1} < 0 < x_{n,1}^{[1,2]} < \varphi_{n,2}$, then D_n does not change of sign in $\left[0, x_{n,1}^{[1,2]}\right]$, thus $P_n^{[1,1]}(0)P_n^{[1,1]}(x_{n,1}^{[1,2]}) > 0$ and as a consequence $x_{n,1}^{[1,2]} < x_{n,1}^{[1,1]}$. Analogously, we can see that, from Lemma 4, and for $i=2,\dots,n-1$,

$$P_n^{[1,1]}\left(x_{n,i}^{[1,2]}\right) P_n^{[1,1]}\left(x_{n,i+1}^{[1,2]}\right) = \frac{\left(\gamma_n^{[1,2]}\right)^2 D_n\left(x_{n,i}^{[1,2]}\right) D_n\left(x_{n,i+1}^{[1,2]}\right)}{\left(x_{n,i}^{[1,2]}-\zeta_1\right)\left(x_{n,i+1}^{[1,2]}-\zeta_1\right)} < 0.$$

It is straightforward show that $P_n^{[1,1]}\left(x_{n,n}^{[1,2]}\right) < 0$. Thus, we complete the proof of the following proposition.

proposition 8 *If $\varphi_{n,1}<0$, then*

$$x_{n,i}^{[1,2]} < x_{n,i}^{[1,1]},$$

for $i=1,\dots,n$.

4. Rational Transformations

Let $\{T_n\}$ and $\{G_n\}$ be the monic sequences of polynomials orthogonal with respect to

$$\langle p,q\rangle = \int_0^\infty p(x)q(x)\frac{(x-\zeta)}{(x-\nu)} d\mu_\omega, \quad \zeta,\nu<0. \tag{23}$$

and

$$\langle p,q\rangle = \int_0^\infty p(x)q(x)\frac{1}{(x-\nu)} d\mu_\omega,\ \nu<0. \tag{24}$$

respectively. Also, let $\|.\|_{\zeta,\nu}$ and $\|.\|_{[\nu]}$ be the the respective induced norms. Again we consider the MOPS $\left\{P_n^{[1]}\right\}$ associated to the weight $(x-\zeta)\omega$ on $[0,\infty)$.

If $(x-\nu)P_n$ is expanded in terms of the basis $\{G_n\}$ we get

$$(x-\nu)P_n(x) = G_n(x) + \sum_{j=0}^{n-1} a_{n,j} G_j(x),$$

where, immediately, $a_{n,j}=0$, $j=1,\dots,n-2$, and

$$a_{n,n-1} = \frac{\int_0^\infty P_n(x)G_n(x)w(x)dx}{\|G_n\|_{[\nu]}^2} = \frac{\|P_n\|^2}{\|G_n\|_{[\nu]}^2}.$$

Thus

$$(x-\nu)P_n(x) = G_{n+1}(x) + \frac{\|P_n\|^2}{\|G_n\|_{[\nu]}^2} G_n(x). \tag{25}$$

Let $\{g_{n,i}\}_{i=1}^n$ be the zeros of G_n arranged in an increasing order. By Lemma 1 applied to (25) we obtain for $i=1,\dots,n$.

$$g_{n,i} < x_{n,i} < g_{n+1,i+1}. \tag{26}$$

Now, we adopt the notation $\{G_n(.,k)\}$ to represent the monic sequence of polynomials orthogonal with respect to

$$\langle p,q\rangle = \int_0^\infty p(x)q(x)\frac{1}{(x-\nu)^k} d\mu, \tag{27}$$

$k\in\mathbb{N}\cup\{0\}$, $\nu<0$. Here, for every n we get $G_n(.,0) := P_n$ and $G_n(.,1) := G_n$. In addition $\{g_{n,i}(k)\}_{i=1}^n$ represents the zeros of $\{G_n(.,k)\}$. As a direct consequence of (26), zeros of $\{G_n(.,k)\}$ and $\{G_n(.,k+1)\}$ are interlaced as follows:

$$g_{n,i}(k+1) < g_{n,i}(k),$$

for $i=1,\dots,n$. In this way, we get the following result.

proposition 9 *For $k,m \in \mathbb{N}\cup\{0\}$, $k<m$, and for $i=1,\ldots,n$*

$$g_{n,i}(m) < g_{n,i}(k).$$

Now, we are going to expand $(x-\zeta)T_n$ by using the basis $\{G_n\}$ and $(x-\nu)P_n^{[1]}$, by using $\{T_n\}$. On the one hand, we get

$$(x-\zeta)\,T_n(x) = G_{n+1}(x) + \frac{\|T_n\|^2_{\zeta,\nu}}{\|G_n\|^2_{[\nu]}} G_n(x),$$

and evaluating in ζ, we have

$$(x-\zeta)\,T_n(x) = G_{n+1}(x) - \frac{G_{n+1}(\zeta)}{G_n(\zeta)} G_n(x). \qquad (28)$$

In [18] the next connection formula is presented:

$$G_n(x) = P_n(x) - \frac{F_n(\nu)}{F_{n-1}(\nu)} P_{n-1}(x), \qquad (29)$$

with

$$F_n(z) = \int_0^\infty \frac{P_n(t)}{z-t}\omega(t)dt, \quad z \in \mathbb{C}\backslash[0,\infty),$$

where the sequence $\{F_n\}$ is very known in the literature and so-called the Cauchy integrals of $\{P_n\}$, or functions of the second kind associated with $\{P_n\}$. In the genesis of what is known today as orthogonal polynomials, the functions of the second kind allowed to find the relationship between continued fractions and orthogonality. The nice work [28] is highly recommended.

In the same way, we obtain

$$(x-\nu)\,P_n^{[1]}(x) = T_{n+1}(x) + \frac{\left\|P_n^{[1]}\right\|_1^2}{\|T_n\|^2_{\zeta,\nu}} T_n(x),$$

and evaluating in ν we get

$$(x-\nu)\,P_n^{[1]}(x) = T_{n+1}(x) - \frac{T_{n+1}(\nu)}{T_n(\nu)} T_n(x). \qquad (30)$$

Now, multiplying (30) by $(x-\zeta)$, and using formula (28), we get

$$\begin{aligned}
&(x-\zeta)(x-\nu)P_n^{[1]}(x) \\
&= (x-\zeta)T_{n+1}(x) - \frac{T_{n+1}(\nu)}{T_n(\nu)}(x-\zeta)T_n(x) \\
&= G_{n+2}(x) - \left(\frac{G_{n+2}(\zeta)}{G_{n+1}(\zeta)} + \frac{T_{n+1}(\nu)}{T_n(\nu)}\right) G_{n+1}(x) \\
&+ \frac{G_{n+1}(\zeta)}{G_n(\zeta)}\frac{T_{n+1}(\nu)}{T_n(\nu)} G_n(x).
\end{aligned}$$

This is a connection formula that allow us to build every polynomial $P_n^{[1]}$ by means of polynomials of the sequence $\{G_n\}$. We summarize in the next proposition.

proposition 10 *For every n*

$$\begin{aligned}
&(x-\zeta)(x-\nu)P_n^{[1]}(x) \\
&= G_{n+2}(x) + \phi_{n,1}(\zeta,\nu)G_{n+1}(x) + \phi_{n,2}(\zeta,\nu)G_n(x),
\end{aligned}$$

where

$$\phi_{n,1}(\zeta,\nu) = -\left(\frac{G_{n+2}(\zeta)}{G_{n+1}(\zeta)} + \frac{T_{n+1}(\nu)}{T_n(\nu)}\right) > 0,$$

and

$$\phi_{n,2}(\zeta,\nu) = \frac{G_{n+1}(\zeta)}{G_n(\zeta)}\frac{T_{n+1}(\nu)}{T_n(\nu)} > 0.$$

Let $\left\{x_{n,i}^{[1]}\right\}_{i=1}^n$ and $\{t_{n,i}\}_{i=1}^n$ be the zeros of $P_n^{[1]}$ and T_n respectively, as before, arranged in an increasing order. Since $-\frac{G_{n+1}(\zeta)}{G_n(\zeta)} > 0$ in (28), by Lemma 1 we obtain

$$g_{n,i} < t_{n,i} < g_{n+1,i+1}, \qquad (31)$$

for $i=1,\ldots,n$. In the same way, since $-\frac{T_{n+1}(\nu)}{T_n(\nu)} > 0$ in (30) we get

$$t_{n,i} < x_{n,i}^{[1]} < t_{n+1,i+1}, \qquad (32)$$

for $i=1,\ldots,n$. Finally, by means of the inequalities (3), (26), (31) and (32) we can prove the following result.

proposition 11 *For every $n \in \mathbb{N}$, the zeros of G_n, T_n and $P_n^{[1]}$ are interlaced as follows:*

$$g_{n,i} < t_{n,i} < x_{n,i}^{[1]},$$

for $i=1,\ldots,n$.

5. Conclusions

In general, we have considered the inner product

$$\langle p,q\rangle = \int_0^\infty p(x)q(x)\omega(x)dx,$$

on the space of real polynomials.

- We have built an appropriate connection formula to prove interlacing of zeros of polynomials of equal degree that belong to sequences of monic polynomials orthogonal with respect to the Christoffel transformations $(x-\zeta_1)\omega$ and $(x-\zeta_2)\omega$, respectively, with $\zeta_1<\zeta_2<0$. As far as we know, the proof is new.
- For different values of k, we have exhibited the natural interlacing of zerosof polynomials of equal degree and orthogonal with respect to Rational transformations as $\frac{1}{(x-\nu)^k}\omega$, $\nu<0$.
- We have considered the canonical transformations $\frac{(x-\zeta)}{(x-\nu)}\omega$, $\frac{1}{(x-\nu)}\omega$ and $(x-\zeta)\omega$, with $\zeta,\nu<0$ and $\zeta \neq \nu$. In this way, we have obtained interlacing properties for zeros of polynomials of equal degree, orthogonal with respect to each of them. As far as we know, the proof is unpublished.

Acknowledgements

The authors thank the anonymous referee for his detailed revision. This review has contributed to improve the presentation of the paper.

Declaration of conflict of interest: The authors declare that they have no conflicts of interest.

Referencias

[1] E. B. Christoffel, "Über die Gaussische Quadratur und eine Verallgemeinerung derselben", Journal für die reine und angewandte Mathematik, vol. 55, pp. 61-82, 1858. http://eudml.org/doc/147722.

[2] T.S. Chihara, An introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.

[3] G. Szegő, Orthogonal Polynomials, 4th ed.American Mathematical Society Colloquim Publication Series, vol. 23, American Mathematical Society, Providence, RI, 1975.

[4] G.J. Yoon, "Darboux transforms and orthogonal polynomials", Bull. Korean Math. Soc. vol. 39, pp. 359–376, 2002. https://doi.org/10.4134/BKMS.2002.39.3.359.

[5] S. Elhay, J. Kautsky, "Jacobi matrices for measures modified by a rational factor", Numer. Algorithms, vol. 6, pp. 205-227, 1994. https://doi.org/10.1007/BF02142672.

[6] B. X. Fejzullahu,R. X. Zejnullahu, "Orthogonal polynomials with respect to the Laguerre measure perturbed by the canonical transformations", Integ. Transf. Spec. Funct. vol. 21, pp. 569-580, 2010. https://doi.org/10.1080/10652460903442032.

[7] E. Huertas, F. Marcellán, B. Xh. Fejnullahu, R.Xh. Zejnullahu, "On orthogonal polynomials with respect to certain discrete Sobolev inner product", Pacific J. Math. vol. 257-1, pp. 167-188, 2012. DOI: 10.2140/pjm.2012.257.167.

[8] E. J. Huertas, F. Marcellán, F. R. Rafaeli, "Zeros of orthogonal polynomials generated by canonical perturbations on measures", Appl. Math. Comput. vol. 218, pp. 7109–7127, 2012. https://doi.org/10.1016/j.amc.2011.12.073.

[9] V. B. Uvarov, "Relation between polynomials orthogonal with different weights", (in Russian), Dokl. Akad. Nauk SSSR vol. 126, pp. 33–36, 1959.

[10] V. B. Uvarov, "The connection between systems of polynomials that are orthogonal with respect to different distributionfunctions", (in Russian), Z. Vycisl. Mat. i Mat. Fiz. vol. 9, pp. 1253–1262, 1969. [English translation in USSR Comput. Math. Math. Phys. vol. 9, pp. 25–36, 1969]. https://doi.org/10.1016/0041-5553(69)90124-4.

[11] A. Zhedanov, "Rational spectral transformations and orthogonal polynomials", J. Comput. Appl. Math. vol. 85, pp. 67–86, 1997. https://doi.org/10.1016/S0377-0427(97)00130-1.

[12] Ya.L. Geronimus, "On the polynomials orthogonal with respect to a given number sequence", Zap. Mat. Otdel. Khar'kov. Univers. i NII Mat. i Mehan. vol. 17, pp. 3-18, 1940.

[13] Ya.L. Geronimus, "On the polynomials orthogonal with respect to a given number sequence and a theorem by W. Hahn", lzv. Akad. Nauk SSSR, vol. 4-2, pp. 215-228, 1940. http://mi.mathnet.ru/eng/izv/v4/i2/p215.

[14] A. Branquinho, F. Marcellán, "Generating new classes of orthogonal polynomials", Int. J. Math. Math. Sci. vol. 19(4), pp. 643–656, 1996. DOI: 10.1155/S0161171296000919.

[15] M.I. Bueno, A. Deaño, E. Tavernetti, "A new algorithm for computing the Geronimus transformation with large shifts", Numer. Alg. vol. 54, pp. 101–139, 2010. https://doi.org/10.1007/s11075-009-9325-9.

[16] M.I. Bueno, F. Marcellán, "Darboux transformations and perturbation of linear functionals", Linear Algebra Appl. vol. 384, pp. 215–242, 2004. https://doi.org/10.1016/j.laa.2004.02.004

[17] M. Derevyagin, F. Marcellán, "A note on the Geronimus transformation and Sobolev orthogonal polynomials", Numer. Algorithms, vol. 67, pp. 1–17, 2013. https://doi.org/10.1007/s11075-013-9788-6.

[18] W. Gautschi, Orthogonal Polynomials: Computation and Approximation. Numerical Mathematics and Scientific Computation Series. Oxford University Press, New York, 2004.

[19] P. Maroni, "Sur la suite de Polynômes Orthogonaux Associée à la forme $u=\delta(c)+\lambda(x-c)^{-1}L$". Period. Math. Hungar. vol. 21(3), pp. 223–248, 1990. https://doi.org/10.1007/bf02651091.

[20] V. Spiridonov, A. Zhedanov, "Discrete-time Volterra chain and classical orthogonal polynomials", J. Phys. A. Math. Gen. vol. 30-24, pp. 8727–8737, 1997. DOI 10.1088/0305-4470/30/24/031.

[21] B. X. Fejzullahu, "Asymptotics for orthogonal polynomials with respect to the Laguerre measure modified by a rational factor", Acta Sci. Math. .Szeged. vol. 77, pp. 73–85, 2011. https://doi.org/10.1007/BF03651367.

[22] D. Galant, "An implementation of Christoffel's theorem in the theory of orthogonal polynomials", Math. Comp. vol. 25, pp. 111–113, 1979. DOI:10.1090/S0025-5718-1971-0288954-8.

[23] D. Galant, “Algebraic methods for modified orthogonal polynomials”, Math. Comp. 59, pp. 541–546, 1992. DOI: https://doi.org/10.1090/S0025-5718-1992-1140648-6.

[24] W. Gautschi, “An algorithmic implementation of the generalized Christoffel theorem”, in: G. Hämmerlin (Ed.), Numerical Integration, Internat. Ser. Numer. Math., vol. 57, Birkhäuser, Basel, pp. 89–106, 1982. https://doi.org/10.1007/978-3-0348-6308-79

[25] C.F. Bracciali, D.K. Dimitrov, A. Sri Ranga, “Chain sequences and symmetric generalized orthogonal polynomials”, J. Comput. Appl. Math. vol.143, pp. 95-106, 2002. https://doi.org/10.1016/S0377-0427(01)00499-X.

[26] C. Brezinski, K. Driver, M. Redivo-Zaglia, “Quasi-orthogonality with applications to some families of classical orthogonal polynomials”, Appl. Numer. Math. vol. 48, pp. 157–168, February 2004. https://doi.org/10.1016/j.apnum.2003.10.001.

[27] J.A. Shohat, “On mechanical quadratures, in particular, with positive coefficients”, Trans. Amer. Math. Soc. vol 42-3, pp. 461–496, November 1937. https://doi.org/10.2307/1989740.

[28] W. Van Assche, “Orthogonal polynomials, associated polynomials and functions of the second kind”, J. Comput. Appl. Math. vol. 37, pp. 237-249, November 1991. https://doi.org/10.1016/0377-0427(91)90121-Y.