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On refined neutrosophic stochastic process

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Abstract

In this paper, we use the notion of crisp stochastic process and refined neutrosophic theory to define the concept of refined neutrosophic stochastic process. This project came to us due to the necessity to study randomness on neutrosophic theory taking the idea that in real world indeterminacy is around it. Besides, we choose refined neutrosophic theory since there are several indeterminacy situations in one problem. Throughout the paper, we present some important results and define the concept of n-dimensional AHisometry for defining some results on refined neutrosophic process.

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1 Introduction

The notion of neutrosophic probability theory was presented by Smarandache [1]; in this text, the author showed a new topic that could be studied in which indeterminacy is different from randomness. For example, let's consider that we are throwing a coin. In classical probability theory, there are two possibilities, but in real life, it is well-known that other aspects can affect the results. This means that there is an indeterminacy that can show that neither of those possibilities can be obtained. In this order, in 2021 Zeina and Hatip [2] introduced the notion of neutrosophic random variable, and in this paper, the authors proved some relevant results. Following that idea, Granados in 2021 [3] presented some new notions on neutrosophic random variables which complemented the original idea of [2]. Besides, in the same year, Granados and Sanabria [4] showed the notion of independence on neutrosophic random variables. Taking into account these notions, many results have been developed by several authors following these ideas, such as neutrosophic continuous distribution [5], discrete distribution [6], and convergence [7]. Furthermore, some real-life applications have been presented. Nguyen et al. [8] used an interval neutrosophic to present a model using stochastic Brownian motion; and Mullai et al. [9] presented an inventory model by applying these concepts.

The concept of refined neutrosophic is one of the most important notions in neutrosophic theory since it admits the existence of a division of indeterminacy. Indeterminacy can be divided into several parts. When indeterminacy is divided into two parts, it is called quadipartitioned; when divided into three parts, it is called pentapartitioned; when divided into four parts, it is called heptapartitioned; and when divided into n-parts, it is called refined indeterminacy [10]. We shall recall that when refined neutrosophic is studied, it is the more general concept that can be studied since all the divisions, from one until n, are involved in that notion. Following this idea, we will present a generalization of neutrosophic stochastic process [11] by using the n-refined concept. Besides, we present the notion of n-dimensional AH-isometry, which is important to prove our theorems. The concept of refined literal indeterminacy neutrosophic number (refined literal neutrosophic number) was defined as $\mathscr{N}_r = a + b_1 I_1 + \ldots + b_n I_n$ for $i \in \mathbb{N}$, where $a, b_1, \ldots, b_n \in \mathbb{R}$.

2 Refined Literal Neutrosophic Stochastic Process

Definition 2.1

Let $\{\zeta(t), t \in T\}$ and $\{\eta_1(t), t \in T\}$, $\{\eta_2(t), t \in T\}$, $\ldots$, $\{\eta_n(t), t \in T\}$ be a collection of crisp (classical) stochastic processes. Refined literal neutrosophic stochastic process is defined as $\{\mathscr{N}_r(t), t \in T\}$ where $\mathscr{N}_r(t) = \zeta(t) + \eta_1(t)I_1 + \eta_2(t)I_2 + \ldots + \eta_n(t)I_n$, $n \in \mathbb{N}$; for $N : (\Omega \times T) \to \mathbb{R}(I)$. $\zeta(t)$ and $\eta_n(t)$ are called the determinant part and indeterminant parts of $\mathscr{N}_r(t)$, respectively.

Remark 2.2

For a better use, we sometimes write $\mathscr{N}_r(t) = \zeta(t) + \sum_{i=1}^{n} \eta_n(t) I_n$.

Theorem 2.3

Let $\{\mathscr{N}_r(t), t \in T\}$ be a refined literal neutrosophic stochastic process. Then, the ensemble average function of $\{\mathscr{N}_r(t), t \in T\}$ is given by:

$$\mu_{\mathscr{N}_r}(t) = \mu_\zeta(t) + \sum_{i=1}^{n} \mu_{\eta_n}(t) I_n.$$

Proof. For a fixed $t \in T$, $\{\zeta(t), t \in T\}$ and $\{\eta_n(t), t \in T\}$, $n \in \mathbb{N}$, become random variables. Thus, $\{\mathscr{N}_r(t), t \in T\}$ becomes a literal neutrosophic random variable. Based on properties of literal neutrosophic random variables, we have:

$$\begin{aligned}
\mu_{\mathscr{N}_r}(t) &= E[\mathscr{N}_r(t)] \\
&= E\left[\zeta(t) + \sum_{i=1}^{n} \eta_n(t) I_n\right] \\
&= E[\zeta(t)] + E\left[\sum_{i=1}^{n} \eta_n(t) I_n\right] \\
&= E[\zeta(t)] + \sum_{i=1}^{n} E[\eta_n(t) I_n] \\
&= \mu_\zeta(t) + \sum_{i=1}^{n} \mu_{\eta_n}(t) I_n.
\end{aligned}$$

Theorem 2.4

Let $\{\mathscr{N}_r(t), t \in T\}$ be a refined literal neutrosophic stochastic process. Then, the autocorrelation function is given by:

$$\mathscr{R}_{\mathscr{N}_r}(s,t) = \mathscr{R}_\zeta(s,t) + \sum_{i=1}^{n} I_n \left[\mathscr{R}_{\zeta\eta_n}(s,t) + \mathscr{R}_{\eta_n\zeta}(s,t) + \mathscr{R}_{\eta_n}(s,t)\right].$$

Proof.

$$\begin{aligned}
\mathscr{R}_{\mathscr{N}_r}(s,t) &= \mathbb{E}[\mathscr{N}_r(s) \cdot \mathscr{N}_r(t)] \\
&= \mathbb{E}\left[\left(\zeta(s) + \sum_{i=1}^{n} \eta_n(s) I_n\right) \cdot \left(\zeta(t) + \sum_{i=1}^{n} \eta_n(t) I_n\right)\right] \\
&= \mathbb{E}\left[\zeta(s)\zeta(t) + \zeta(s)\sum_{i=1}^{n} \eta_n(t) I_n + \sum_{i=1}^{n} \eta_n(s) I_n \zeta(t) + \sum_{i=1}^{n} \eta_n(s) I_n \sum_{i=1}^{n} \eta_n(t) I_n\right] \\
&= \mathbb{E}\left[\zeta(s)\zeta(t) + \zeta(s)\sum_{i=1}^{n} \eta_n(t) I_n + \sum_{i=1}^{n} \eta_n(s) I_n \zeta(t) + \sum_{i=1}^{n} \eta_n(s)\eta_n(t) I_n\right] \\
&= \mathbb{E}[\zeta(s)\zeta(t)] + \mathbb{E}\left[\sum_{i=1}^{n} \eta_n(s) I_n \zeta(t) + \zeta(s)\sum_{i=1}^{n} \eta_n(t) I_n + \sum_{i=1}^{n} \eta_n(s)\eta_n(t) I_n\right] \\
&= \mathscr{R}_\zeta(s,t) + \sum_{i=1}^{n} I_n \left[\mathscr{R}_{\zeta\eta_n}(s,t) + \mathscr{R}_{\eta_n\zeta}(s,t) + \mathscr{R}_{\eta_n}(s,t)\right].
\end{aligned}$$

Remark 2.5

If $s = t$, $\mathscr{R}_{\mathscr{N}_r}(t,t)$ is followed by

$$\mathscr{R}_\zeta(t,t) + \sum_{i=1}^{n} I_n \left[\mathscr{R}_{\zeta\eta_n}(t,t) + \mathscr{R}_{\eta_n\zeta}(t,t) + \mathscr{R}_{\eta_n}(t,t)\right] =$$

$$E\left[\zeta^2(t)\right] + \sum_{i=1}^{n} I_n \left[2\mathscr{R}_{\zeta\eta_n}(t,t) + E\left[\eta_n^2(t)\right]\right].$$

Theorem 2.6
Let $\{\mathscr{N}_r(t), t \in T\}$ be a refined literal neutrosophic stochastic process. Then, its auto-covariance function is given by

$$\mathscr{C}_{\mathscr{N}_r}(s,t) = \mathscr{R}_{\mathscr{N}_r}(s,t) - \mu_{\mathscr{N}_r}(s)\mu_{\mathscr{N}_r}(t).$$

Proof.

$$\begin{aligned}\mathscr{C}_{\mathscr{N}_r}(s,t) &= \mathrm{cov}[\mathscr{N}_r(s), \mathscr{N}_r(t)] = E\{[\mathscr{N}_r(s) - \mu_{\mathscr{N}_r}(s)][\mathscr{N}_r(t) - \mu_{\mathscr{N}_r}(t)]\}\\ &= E\{\mathscr{N}_r(s)\mathscr{N}_r(t) - \mu_{\mathscr{N}_r}(t)\mathscr{N}_r(s) - \mu_{\mathscr{N}_r}(s)\mathscr{N}_r(t) + \mu_{\mathscr{N}_r}(s)\mu_{\mathscr{N}_r}(t)\}\\ &= \mathscr{R}_{\mathscr{N}_r}(s,t) - \mu_{\mathscr{N}_r}(t)\mu_{\mathscr{N}_r}(s) - \mu_{\mathscr{N}_r}(s)\mu_{\mathscr{N}_r}(t) + \mu_{\mathscr{N}_r}(s)\mu_{\mathscr{N}_r}(t)\\ &= \mathscr{R}_{\mathscr{N}_r}(s,t) - \mu_{\mathscr{N}_r}(s)\mu_{\mathscr{N}_r}(t).\end{aligned}$$

i.e,

$$\begin{aligned}\mathscr{C}_{\mathscr{N}_r}(s,t) &= \mathscr{R}_{\zeta}(s,t) + \sum_{i=1}^{n} I_n \left[\mathscr{R}_{\zeta\eta_n}(s,t) + \mathscr{R}_{\eta_n\zeta}(s,t) + \mathscr{R}_{\eta_n}(s,t)\right]\\ &\quad - \left(\mu_{\zeta}(s) + \sum_{i=1}^{n} \mu_{\eta_n}(s) I_n\right)\left(\mu_{\zeta}(t) + \sum_{i=1}^{n} \mu_{\eta_n}(t) I_n\right)\\ &= \mathscr{R}_{\zeta}(s,t) - \mu_{\zeta}(s)\mu_{\zeta}(t) + \sum_{i=1}^{n} I_n[\mathscr{R}_{\zeta\eta_n}(s,t) +\\ &\qquad \mathscr{R}_{\eta_n\zeta}(s,t) + \mathscr{R}_{\eta_n}(s,t)]\\ &\quad - \sum_{i=1}^{n} I_n \left[\mu_{\eta_n}(s)\mu_{\zeta}(t) + \mu_{\zeta}(s)\mu_{\eta_n}(t) + \mu_{\eta_n}(s)\mu_{\eta_n}(t)\right]\\ &= \mathscr{R}_{\zeta}(s,t) - \mu_{\zeta}(s)\mu_{\zeta}(t) + \sum_{i=1}^{n} I_n \left[\mathscr{R}_{\zeta\eta_n}(s,t) + \mathscr{R}_{\eta_n\zeta}(s,t) +\right.\\ &\qquad \left.\mathscr{R}_{\eta_n}(s,t) - \mu_{\eta_n}(s)\mu_{\zeta}(t) - \mu_{\zeta}(s)\mu_{\eta_n}(t) - \mu_{\eta_n}(s)\mu_{\eta_n}(t)\right].\end{aligned}$$

Remark 2.7. If $s = t$, $\mathscr{C}_{\mathscr{N}_r}(s,t)$ is followed by

$$\begin{aligned}\mathrm{cov}[\mathscr{N}_r(t), \mathscr{N}_r(t)] &= E\{[\mathscr{N}_r(t) - \mu_{\mathscr{N}_r}(t)][\mathscr{N}_r(t) - \mu_{\mathscr{N}_r}(t)]\}\\ &= E[(\mathscr{N}_r(t) - \mu_{\mathscr{N}_r}(t))^2]\\ &= \mathrm{var}[\mathscr{N}_r(t)].\end{aligned}$$

In this case, $\mathrm{var}[\mathscr{N}_r(t)]$ is given by

$$\begin{aligned}&\mathscr{R}_{\zeta}(t,t) - \mu_{\zeta}(t)\mu_{\zeta}(t) + \sum_{i=1}^{n} I_n \left[\mathscr{R}_{\zeta\eta_n}(t,t) + \mathscr{R}_{\eta_n\zeta}(t,t) + \mathscr{R}_{\eta_n}(t,t)\right.\\ &\qquad \left.-\mu_{\eta_n}(t)\mu_{\zeta}(t) - \mu_{\zeta}(t)\mu_{\eta_n}(t) - \mu_{\eta_n}(t)\mu_{\eta_n}(t)\right]\\ &= E\left[\zeta^2(t)\right] - \mu_{\zeta}^2(t) + \sum_{i=1}^{n} I_n \left[2\mathscr{R}_{\zeta\eta_n}(t,t)\right.\\ &\qquad \left.-2\mu_{\zeta}(t)\mu_{\eta_n}(t) - \mu_{\eta_n}^2(t) + E\left[\eta_n^2(t)\right]\right].\end{aligned}$$

Theorem 2.8. Let $\mathscr{N}_r(t_1) = \zeta(t_1) + \sum_{i=1}^{n} \eta_n(t_1) I_n$ and $M_r(t_2) = \omega(t_2) + \sum_{j=1}^{m} \rho_m(t_2) I_m$ be two refined literal neutrosophic stochastic processes where $\zeta(t_1)$, $\eta_n(t_1)$, $\omega(t_2)$, and $\rho_m(t_2)$ are independent. Then,

$$\begin{aligned}E[\mathscr{N}_r(t_1)M_r(t_2)] &= E\left[\left(\zeta(t_1) + \sum_{i=1}^{n} \eta_n(t_1) I_n\right)\left(\omega(t_2) + \sum_{j=1}^{m} \rho_m(t_2) I_m\right)\right]\\ &= E[\zeta(t_1)\omega(t_2)] + E\left[\zeta(t_1)\sum_{j=1}^{m} \rho_m(t_2) I_m\right]\\ &\quad + E\left[\omega(t_2)\sum_{i=1}^{n} \eta_n(t_1) I_n\right] + E\left[\sum_{i=1}^{n} \eta_n(t_1) I_n \sum_{j=1}^{m} \rho_m(t_2) I_m\right].\end{aligned}$$

Since $\zeta(t_1)$, $\eta_n(t_1)$, $\omega(t_2)$ and $\rho_m(t_2)$ are independent, we have

$$\begin{aligned}E[\mathscr{N}_r(t_1)M_r(t_2)] &= E[\zeta(t_1)] + E[\omega(t_2)] + E[\zeta(t_1)] + E\left[\sum_{j=1}^{m} \rho_m(t_2) I_m\right]\\ &\quad + E[\omega(t_2)] + E\left[\sum_{i=1}^{n} \eta_n(t_1) I_n\right] + E\left[\sum_{i=1}^{n} \eta_n(t_1) I_n\right] +\\ &\quad E\left[\sum_{j=1}^{m} \rho_m(t_2) I_m\right]\\ &= E\left[\zeta(t_1) + \sum_{i=1}^{n} \eta_n(t_1) I_n\right] E\left[\omega(t_2) + \sum_{j=1}^{m} \rho_m(t_2) I_m\right]\\ &= E[\mathscr{N}_r(t_1)]E[M_r(t_2)].\end{aligned}$$

It is well-know that in neutrosophic literature, AH-Isometry is an important tool that can help to results some problems. However, refined AH-Isometry has not been studied in general cases so far. For that reason, before to present some results on refined literal neutrosophic stochastic process, we should first to prove some results on refined AH-Isometry.

We know that I can be split into several parts as needed, i.e., $I_1, I_2, \ldots, I_n$ with conditions $I_m^n = I_n$ for $n, m \in \mathbb{N}$. Besides, $I_n I_{n-1} I_{n-2} \ldots I_1 = I_1 I_2 \ldots I_{n-1} I_n = I_1$.

Definition 2.9 Let $(R, +, \times)$ be a ring. $(R(I_1, \ldots, I_n), +, \times)$ is called a refined neutrosophic ring generated by $R, I_1, \ldots, I_n$.

Definition 2.10 The refined n-dimensional AH-isometry between the ring $R(I_1, \ldots, I_n)$ and the Cartesian product $R \times R \times \ldots \times R$ is defined as follows:

$$g : R(I_1, \ldots, I_n) \to R \times R \times \ldots \times R$$

$$g(a + b_1 I_1 + \ldots + b_n I_n) = (a, a + b_1 + b_2 + \ldots + b_n, a + b_2 + \ldots + b_n, \ldots, a + b_n)$$

Example 2.11 For the ring $(\mathbb{Z}, +, \times)$, the corresponding refined n-dimensional neutrosophic ring is

$$\mathbb{Z}(I) = \left(m + \sum_{j=1}^{p} n_p I_p; m, n_p \in \mathbb{Z} \text{ and } p \in \mathbb{N}\right).$$

Theorem 2.12 Let R be any ring with unity 1, $R(I_1, \ldots, I_n)$ be its corresponding refined n-dimensional neutrosophic ring. Then, $R(I_1, \ldots, I_n) \cong R \times R \times \ldots \times R$.

Proof. We can present the refined n-dimensional AH-Isometry between $R(I_1, \ldots, I_n)$ and $R \times R \times \ldots \times R$ as

$$\begin{gathered}g : R(I_1, \ldots, I_n) \to R \times R \times \ldots \times R\\ g(a + b_1 I_1 + \ldots + b_n I_n) = (a, a + b_1 + b_2 + \ldots +\\ [b_n, a + b_2 + \ldots + b_n, \ldots, a + b_n).\end{gathered}$$

Let $t_1 = a + b_1 I_1 + \ldots + b_n I_n$ and $t_2 = c + d_1 I_1 + \ldots + d_n I_n$ be two refined n-dimensional neutrosophic elements. Then,

$$\begin{aligned} g(t_1+t_2) &= g([a+c]+[b_1+d_1]I_1+\ldots+[b_n+d_n]I_n) \\ &= (a+c, a+c+b_1+d_1+b_2+d_2+\ldots+b_n+d_n, a+c+b_2+d_2+ \\ &\quad \ldots+b_n+d_n,\ldots,a+c+b_n+d_n) \\ &= (a, a+b_1+b_2+\ldots+b_n,\ldots,a+b_n)+(c,c+d_1+d_2+\ldots+ \\ &\quad d_n,\ldots,c+d_n) \\ &= g(a+b_1I_1+b_2I_2+\ldots+b_nI_n)+g(c+d_1I_1+d_2I_2+\ldots+d_nI_n) \\ &= g(t_1)+g(t_2). \end{aligned}$$

Now,

$$\begin{aligned} g(t_1\cdot t_2) &= g((a+b_1I_1+\ldots+b_nI_n)\cdot(c+d_1I_1+\ldots+d_nI_n)) \\ &= g\left(a+\sum_{j=1}^{p} b_pI_p\right)\cdot\left(c+\sum_{j=1}^{p} d_pI_p\right) \\ &= g\left(ac+a\sum_{j=1}^{p} d_pI_p+c\sum_{j=1}^{p} b_pI_p+\sum_{j=1}^{p} b_pd_pI_p\right) \\ &= g\left(ac+\sum_{j=1}^{p} I_p(ad_p+cb_p+b_pd_p)\right) \\ &= \left(ac, ac+\sum_{j=1}^{p}(ad_p+cb_p+b_pd_p), ac+\sum_{j=2}^{p}(ad_p+cb_p+b_pd_p),\right. \\ &= \quad \left.\ldots, ac+ad_p+cb_p+b_pd_p\right) \\ &= \left(a, a+\sum_{j=1}^{p} b_p,\ldots,a+b_p\right)\cdot\left(c,c+\sum_{j=1}^{p} d_p,\ldots,c+d_p\right) \\ &= g(a+b_1I_1+\ldots+b_nI_n)\cdot g(c+d_1I_1+\ldots+d_nI_n) \\ &= g(t_1)\cdot g(t_2). \end{aligned}$$

Therefore, g is a correspondence one to one, due to $\ker(g)=\{0\}$, and for every $(a,b_1,b_2,\ldots,b_n)\in\mathbb{R}\times\mathbb{R}\times\ldots\times\mathbb{R}$, there exists $x=a+(b_1-b_2)I_1+(b_2-b_3)I_2+\ldots+(b_n-a)I_n\in\mathbb{R}(I_1,\ldots,I_n)$, such that $g(x)=(a,b_1,b_2,\ldots,b_n)$. Hence, g is an isomorphism.

Remark 2.13. The inverse isomorphism of g is $g^{-1}:\mathbb{R}\times\mathbb{R}\times\ldots\times\mathbb{R}\to\mathbb{R}(I_1,I_2,\ldots,I_n)$ with $g^{-1}(u_1,u_2,\ldots,u_n)=(u_1,(u_2-u_1)I_1,(u_3-u_1)I_2,\ldots,(u_n-u_1)I_n)$.

With the previous theorem, it can be easy to define and prove results on refined literal neutrosophic stochastic processes using AH-isometry.

Definition 2.14. Let $\{\mathscr{N}_r(t), t\in T\}$ be a refined literal neutrosophic stochastic process. Applying AH-isometry on $\mathscr{N}_r(t), t\in T$ yields to

$$\begin{aligned} g[\mathscr{N}_r(t)] &= g\left(\zeta(t)+\sum_{i=1}^{n}\eta_n(t)I_n\right) \\ &= \left(\zeta(t),\ \zeta(t)+\sum_{i=1}^{n}\eta_n(t),\ \zeta(t)+\sum_{i=2}^{n}\eta_n(t),\ldots,\ \zeta(t)+\eta_n(t)\right). \end{aligned}$$

Notice that using refined n-dimensional AH-isometry, we transfer the refined literal neutrosophic stochastic process $\{\mathscr{N}_r(t), t\in T\}$ into n-classical stochastic processes:

$$\{\zeta(t), t\in T\}, \left(\zeta(t)+\sum_{i=1}^{n}\eta_n(t), t\in T\right), \left(\zeta(t)+\sum_{i=2}^{n}\eta_n(t), t\in T\right),$$
$$\ldots, \{\zeta(t)+\eta_n(t), t\in T\}.$$

This means that we can study the characteristics of $\{\mathscr{N}_r(t), t\in T\}$ by studying all its n-classical stochastic processes.

Example 2.15. In theorem 2.3, we proved that $\mu_{\mathscr{N}_r}(t)=\mu_\zeta(t)+\mu_{\eta_n}I_n$. We can obtain the same results by using the n-dimensional AH-isometry as can be seen next:

We have $\mathscr{N}_r(t)=\zeta(t)+\sum_{i=1}^{n}\eta_n(t)I_n$. Thus,

$$E[\mathscr{N}_r(t)] = E\left[\zeta(t)+\sum_{i=1}^{n}\eta_n(t)I_n\right].$$

Applying the isometry function g, we get:

$$g(E[\mathscr{N}_r(t)]) = g\left(E\left[\zeta(t)+\sum_{i=1}^{n}\eta_n(t)I_n\right]\right) = E\left[g\left(\zeta(t)+\sum_{i=1}^{n}\eta_n(t)I_n\right)\right].$$

Thus,

$$E\left[\zeta(t),\zeta(t)+\sum_{i=1}^{n}\eta_n(t)\right] = (\mu_\zeta(t),\mu_\zeta(t)+\mu_{\eta_n}(t)).$$

Using the inverse isometry we get:

$$g^{-1}\left(g(E[\mathscr{N}_r(t)])\right) = E[\mathscr{N}_r(t)] = \mu_\zeta(t)+\mu_{\eta_n}I_n.$$

Definition 2.16. Let $\{\mathscr{N}_r(t), t\in T\}$ be a refined literal neutrosophic stochastic process. We will call $F(x_{\mathscr{N}_r}, t)=P(\mathscr{N}_r(t)\le x_{\mathscr{N}_r})$ the first order distribution of $\{\mathscr{N}_r(t), t\in T\}$ where $x_{\mathscr{N}_r}=x_1+y_1I_1+y_2I_2+\cdots+y_nI_n$ and $x, y_n\in\mathbb{R}$.

Definition 2.17. A refined literal neutrosophic stochastic process is said to be weakly stationary if the following conditions hold:

1. $\mu_{\mathscr{N}_r}(t)=\mu_{\mathscr{N}_r}=\mu_1+\mu_2I_1+\cdots+\mu_nI_{n-1}$.
2. $E[\mathscr{N}_r(t)\cdot\mathscr{N}_r(t-\tau_{\mathscr{N}_r})]=\mathscr{R}_{\mathscr{N}_r}(\tau)$.

Theorem 2.18. A refined literal neutrosophic stochastic process $\mathscr{N}_r(t)=\zeta(t)+\sum_{i=1}^{n}\eta_nI_n$ is weakly stationary if and only if $\{\zeta(t), t\in T\}$, $\left\{\zeta(t)+\sum_{i=1}^{n}\eta_n(t), t\in T\right\}$, $\left\{\zeta(t)+\sum_{i=2}^{n}\eta_n(t), t\in T\right\}$, …, $\{\zeta(t)+\eta_n(t), t\in T\}$ are weakly stationary.

Proof. Let's consider that $\{\zeta(t), t\in T\}$, $\left\{\zeta(t)+\sum_{i=1}^{n}\eta_n(t), t\in T\right\}$, $\left\{\zeta(t)+\sum_{i=2}^{n}\eta_n(t), t\in T\right\}$, …, $\{\zeta(t)+\eta_n(t), t\in T\}$ are weakly stationary. Since $\{\zeta(t), t\in T\}$ is weakly stationary, then $\mu_\zeta(t)=\mu_\zeta=k$, where $k\in\mathbb{R}$ and $E[\zeta(t)\cdot\zeta(t-\tau)]=\mathscr{R}_\zeta(\tau)$. Now, consider that $\left\{\zeta(t)+\sum_{i=1}^{n}\eta_n(t), t\in T\right\}$

is weakly stationary, thus $\mu_{\zeta+\sum_{i=1}^{n}\eta_n(t)} = E\left[\zeta(t)+\sum_{i=1}^{n}\eta_n(t)\right] = \mu_\zeta + \mu_{\eta_n} = t$, where $t \in \mathbb{R}$, this implies $\mu_{\eta_n} = \mu_{\eta_n} = i$, where $i \in \mathbb{R}$, and

$$\begin{aligned}
&\mathscr{R}_{\zeta+\sum_{i=1}^{n}\eta_n}(t,t-\tau)\\
&= E\left[\left(\zeta(t)+\sum_{i=1}^{n}\eta_n(t)\right)\left(\zeta(t-\tau)+\sum_{i=1}^{n}\eta_n(t-\tau)\right)\right]\\
&= E\Bigg[\zeta(t)\zeta(t-\tau)+\zeta(t)\sum_{i=1}^{n}\eta_n(t-\tau)+\sum_{i=1}^{n}\eta_n(t)\zeta(t-\tau)+\\
&\qquad\qquad \sum_{i=1}^{n}\eta_n(t)\eta_n(t-\tau)\Bigg]\\
&= \mathscr{R}_\zeta(t,t-\tau)+\mathscr{R}_{\zeta\eta_n}(t,t-\tau)+\mathscr{R}_{\eta_n\zeta}(t,t-\tau)+\mathscr{R}_{\eta_n}(t,t-\tau)
\end{aligned}$$

Since $\zeta(t)+\sum_{i=1}^{n}\eta_n(t)$ is weakly stationary, then $\mathscr{R}_{\zeta+\sum_{i=1}^{n}\eta_n}(t,t-\tau)$ must only depend on the difference τ, hence the only way of it can be

$$\begin{aligned}
\mathscr{R}_{\zeta+\sum_{i=1}^{n}\eta_n}(t,t-\tau) &= \mathscr{R}_\zeta(t,t-\tau)+\sum_{i=1}^{n}\Bigg[\mathscr{R}_{\zeta\eta_n}(t,t-\tau)+\\
&\qquad \mathscr{R}_{\eta_n\zeta}(t,t-\tau)+\mathscr{R}_{\eta_n}(t,t-\tau)\Bigg]\\
&= \mathscr{R}_{\zeta+\sum_{i=1}^{n}\eta_n}(\tau),
\end{aligned}$$

which means that $\mathscr{R}_{\zeta\eta_n}(t,t-\tau) = \mathscr{R}_{\eta_n}(T)$. Therefore,

$$\begin{aligned}
E[\mathscr{N}_r(t)] &= E\left[\zeta(t)+\sum_{i=1}^{n}\eta_n(t)I_n\right]\\
&= \mu_\zeta(t)+\mu_{\eta_n}(t)I_n\\
&= \mu_\zeta+\mu_{\eta_n}I\\
&= \mu_{\mathscr{N}_r} = a, \text{ where } a \in \mathbb{R}.
\end{aligned}$$

By theorem (2.4),

$$\begin{aligned}
\mathscr{R}_{\mathscr{N}_r}(t,t-\tau) &= \mathscr{R}_\zeta(\tau)+\sum_{i=1}^{n}I_n\left[\mathscr{R}_{\zeta\eta_n}(\tau)+\mathscr{R}_{\eta_n\zeta}(\tau)+\mathscr{R}_{\eta_n}(\tau)\right]\\
&= \mathscr{R}_{\mathscr{N}_r}(\tau).
\end{aligned}$$

The cases for $\left(\sum_{i=2}^{n}\eta_n(t),\, t\in T\right)$, $\left\{\sum_{i=3}^{n}\eta_n(t),\, t\in T\right\}, \ldots,$ $\left\{\sum_{i=n-1}^{n}\eta_n(t),\, t\in T\right\}$ are proved similarly.

Next, we will consider that $\{\zeta(t)+\eta_n(t),\, t\in T\}$ is weakly stationary, thus $\mu_{\zeta+\sum\eta_n(t)} = E[\zeta(t)+\eta_n(t)] = \mu_{\zeta+\eta_n(t)}] = g$, where $g \in \mathbb{R}$, this means $\mu_{\eta_n}(t) = \mu_{\eta_n} = p$, where $p \in \mathbb{R}$, and

$$\begin{aligned}
\mathscr{R}_{\zeta+\eta_n}(t,t-\tau) &= E\left[(\zeta(t)+\eta_n(t))(\zeta(t-\tau)+\eta_n(t-\tau))\right]\\
&= E\Bigg[\zeta(t)\zeta(t-\tau)+\zeta(t)\eta_n(t-\tau)+\\
&\qquad \eta_n(t)\zeta(t-\tau)+\eta_n(t)\eta_n(t-\tau)\Bigg]\\
&= \mathscr{R}_\zeta(t,t-\tau)+\mathscr{R}_{\zeta\eta_n}(t,t-\tau)+\mathscr{R}_{\eta_n\zeta}(t,t-\tau)+\mathscr{R}_{\eta_n}(t,t-\tau)
\end{aligned}$$

Since $\zeta(t)+\eta_n(t)$ is weakly stationary, then $\mathscr{R}_{\zeta+\eta_n}(t,t-\tau)$ must only depend on the difference τ, hence the only way of it can be

$$\begin{aligned}
\mathscr{R}_{\zeta+\eta_n}(t,t-\tau) &= \mathscr{R}_\zeta(t,t-\tau)+\mathscr{R}_{\zeta\eta_n}(t,t-\tau)+\mathscr{R}_{\eta_n\zeta}(t,t-\tau)+\mathscr{R}_{\eta_n}(t,t-\tau)\\
&= \mathscr{R}_{\zeta+\eta_n}(\tau)
\end{aligned}$$

This implies $\mathscr{R}_{\zeta\eta_n}(t,t-\tau) = \mathscr{R}_{\eta_n}(\tau)$. Therefore,

$$\begin{aligned}
E[\mathscr{N}_r(t)] &= E[\zeta(t)+\eta_n(t)I_n]\\
&= \mu_\zeta(t)+\mu_{\eta_n}(t)I_n\\
&= \mu_\zeta+\mu_{\eta_n}I\\
&= \mu_{\mathscr{N}_r}\\
&= s, \text{ where } s \in \mathbb{R}.
\end{aligned}$$

By theorem (2.4),

$$\begin{aligned}
\mathscr{R}_{\mathscr{N}_r}(t,t-\tau) &= \mathscr{R}_\zeta(\tau)+I_n[\mathscr{R}_{\zeta\eta_n}(\tau)+\mathscr{R}_{\eta_n\zeta}(\tau)+\mathscr{R}_{\eta_n}(\tau)]\\
&= \mathscr{R}_{\mathscr{N}_r}(\tau).
\end{aligned}$$

Therefore, we have proved that $\{\mathscr{N}_r(t), t\in T\}$ is weakly stationary. Next, let's assume that $\{\mathscr{N}_r(t), t\in T\}$ is weakly stationary. Since $\{\mathscr{N}_r(t), t\in T\}$ is weakly stationary, then $E[\mathscr{N}_r(t)] = \mu_{\mathscr{N}_r}(t) = \mu_{\mathscr{N}_r} = j$, where $j \in \mathbb{R}$, but $E[\mathscr{N}_r(t)] = \mu_\zeta(t)+\sum_{i=1}^{n}\mu_{\eta_n}(t)I_n$, so both $\mu_\zeta(t)$ and $\mu_{\eta_n}(t)$ must only depend on time. Therefore, $\mu_\zeta = \mu_\zeta(t)$ and $\mu_{\eta_n}(t) = \mu_{\eta_n}$, which implies that $\mu_{\zeta+\eta_n}(t) = \mu_\zeta+\mu_{\eta_n} = u$, where $u \in \mathbb{R}$. Besides, we obtain

$$\mathscr{R}_{\mathscr{N}_r}(t,t-\tau) = \mathscr{R}_\zeta(t,t-\tau)+\sum_{i=1}^{n}I_n[\mathscr{R}_{\zeta\eta_n}(t,t-\tau)+\mathscr{R}_{\eta_n\zeta}(t,t-\tau)+\mathscr{R}_{\eta_n}(t,t-\tau)].$$

And since $\{\mathscr{N}_r(t), t\in T\}$ is weakly stationary, then $\mathscr{R}_{\mathscr{N}_r}(t,t-\tau)$ must only depend on the difference τ. Thus, the following equations hold:

$$\mathscr{R}_\zeta(t,t-\tau) = \mathscr{R}_\zeta(\tau). \tag{2.1}$$

$$\mathscr{R}_{\zeta\eta_n}(t,t-\tau) = \mathscr{R}_{\zeta\eta_n}(\tau), \tag{2.2}$$

$$\mathscr{R}_{\eta_n\zeta}(t,t-\tau) = \mathscr{R}_{\eta_n\zeta}(\tau), \tag{2.3}$$
$$\mathscr{R}_{\eta_n}(t,t-\tau) = \mathscr{R}_{\eta_n}(\tau) \tag{2.4}$$

And since $\mu_\zeta = \mu_\zeta(t)$ and (2.1), we imply that $\{\zeta(t), t\in T\}$ is weakly stationary, and by using (2.2), (2.3), (2.4) and $\mu_{\zeta+\sum\eta_n}(t) = \mu_\zeta+\mu_{\eta_n} = u$, $\left\{\zeta(t)+\sum_{i=1}^{n}\eta_n(t),\, t\in T\right\}$, $\left\{\zeta(t)+\sum_{i=2}^{n}\eta_n(t),\, t\in T\right\}, \ldots, \{\zeta(t)+\eta_n(t),\, t\in T\}$

are weakly stationary.

Theorem 2.19. Consider that $\{\mathcal{N}_r(t), t \in T\}$ is a weakly stationary refined literal neutrosophic process with autocorrelation function $\mathcal{R}_{\mathcal{N}_r}(t)$. Then, the following statements hold:

(1) $\mathcal{R}_{\mathcal{N}_r}(t) = \mathcal{R}_{\mathcal{N}_r}(-\tau)$.

(2) $|\mathcal{R}_{\mathcal{N}_r}(t)| \leq \mathcal{R}_{\mathcal{N}_r}(0)$.

Proof. **(1)** We get

$$\mathcal{R}_{\mathcal{N}_r}(\tau) = \mathcal{R}_{\zeta}(\tau) + \sum_{i=1}^{n} I_n[\mathcal{R}_{\zeta\eta_n}(\tau) + \mathcal{R}_{\eta_n\zeta}(\tau) + \mathcal{R}_{\eta_n}(\tau)].$$

Similarly,

$$\mathcal{R}_{\mathcal{N}_r}(-\tau) = \mathcal{R}_{\zeta}(-\tau) + \sum_{i=1}^{n} I_n[\mathcal{R}_{\zeta\eta_n}(-\tau) + \mathcal{R}_{\eta_n\zeta}(-\tau) + \mathcal{R}_{\eta_n}(-\tau)].$$

Using properties of cross-correlation function in classical stationary processes, we obtain $\mathcal{R}_{\mathcal{N}_r}(t) = \mathcal{R}_{\mathcal{N}_r}(-\tau)$.

(2) Taking n-dimensional AH-isometry,

$$\begin{aligned}
g(|\mathcal{R}_{\mathcal{N}_r}(t)|) &= |E\left[g(\mathcal{N}_r(t)\cdot\mathcal{N}_r(t-\tau))\right]| \\
&= \left|E\left[g\left(\zeta(t) + \sum_{i=1}^{n}\eta_n(t)I_n\right)\left(\zeta(t-\tau) + \sum_{i=1}^{n}\eta_n(t-\tau)I_n\right)\right]\right| \\
&= \left|E\left[g\left(\zeta(t) + \sum_{i=1}^{n}\eta_n(t)I_n\right)g\left(\zeta(t-\tau) + \sum_{i=1}^{n}\eta_n(t-\tau)I_n\right)\right]\right| \\
&= \left|E\left[\left(\zeta(t), \zeta(t) + \sum_{i=1}^{n}\eta_n(t), \ldots, \zeta(t) + \eta_n(t-\tau)\right)\right.\right. \\
&= \left.\left.\quad\left(\zeta(t-\tau), \zeta(t-\tau) + \sum_{i=1}^{n}\eta_n(t-\tau), \ldots, \zeta(t-\tau) + \eta_n(t-\tau)\right)\right]\right| \\
&= (|\mathcal{R}_{\zeta}(\tau)|, \ldots, |\mathcal{R}_{\zeta+\eta_n}(\tau)|) \leq (0, \ldots, 0).
\end{aligned}$$

Now, taking g^{-1}, we get that $|\mathcal{R}_{\mathcal{N}_r}(t)| \leq \mathcal{R}_{\mathcal{N}_r}(0)$.

Definition 2.20. Let $\{\mathcal{N}_r(t), t \in T\}$ and $\{M_r(t), t \in T\}$ be two literal neutrosophic stochastic processes. They are said to be refined jointly weakly stationary if the following conditions hold:

(1) $\mathcal{N}_r(t)$ and $M_r(t)$ are weakly stationary.

(2) $\mathcal{R}_{\mathcal{N}_r M_r}(t, t-\tau) = \mathcal{R}_{\mathcal{N}_r M_r}(\tau)$.

Theorem 2.21. Let $\{\mathcal{N}_r(t), t \in T\}$ and $\{M_r(t), t \in T\}$ be refined jointly weakly stationary literal neutrosophic stochastic processes. Assume the literal neutrosophic stochastic process $Z_r(t)$ defined as $Z_r(t) = \mathcal{N}_r(t) + M_r(t)$. Then, $Z_r(t)$ is weakly stationary.

Proof. Since $\{\mathcal{N}_r(t), t \in T\}$ and $\{M_r(t), t \in T\}$ are refined jointly weakly stationary literal neutrosophic stochastic processes, we have $\mu_{\mathcal{N}_r}(t) = \mu_{\mathcal{N}_r}$ and $\mu_{M_r}(t) = \mu_{M_r}$. Therefore, $E[Z_r(t)] = \mu_{Z_r}(t) = \mu_{M_r}(t) + \mu_{\mathcal{N}_r}(t) = \mu_{M_r} + \mu_{\mathcal{N}_r} = \mu_{Z_r}$.

Now, we know that $\mathcal{R}_{\mathcal{N}_r}(t, t-\tau) = \mathcal{R}_{\mathcal{N}_r}(\tau)$, $\mathcal{R}_{M_r}(t, t-\tau) = \mathcal{R}_{M_r}(\tau)$, $\mathcal{R}_{\mathcal{N}_r M_r}(t, t-\tau) = \mathcal{R}_{\mathcal{N}_r M_r}(\tau)$, and $\mathcal{R}_{M_r\mathcal{N}_r}(t, t-\tau) = \mathcal{R}_{M_r\mathcal{N}_r}(\tau)$, thus

$$\begin{aligned}
\mathcal{R}_{Z_r}(t, t-\tau) &= E[(\mathcal{N}_r(t) + M_r(t))(\mathcal{N}_r(t-\tau) + M_r(t-\tau))] \\
&= E[\mathcal{N}_r(t)\mathcal{N}_r(t-\tau)] + E[M_r(t)\mathcal{N}_r(t-\tau)] \\
&\quad + E[\mathcal{N}_r(t)M_r(t-\tau)] + E[M_r(t)M_r(t-\tau)] \\
&= \mathcal{R}_{\mathcal{N}_r}(\tau) + \mathcal{R}_{M_r\mathcal{N}_r}(\tau) + \mathcal{R}_{\mathcal{N}_r M_r}(\tau) + \mathcal{R}_{M_r}(\tau) \\
&= \mathcal{R}_{Z_r}(\tau).
\end{aligned}$$

Therefore, this proves that $Z_r(t)$ is weakly stationary.

The following example shows a refined neutrosophic stochastic process which is weakly stationary process

Example 2.22. Let $\{\mathcal{N}_r(t), t \in T\}$ be a refined neutrosophic stochastic process defined as follows:

$$\mathcal{N}_r(t) = \zeta(t) + \zeta(t)I_1 + \zeta(t)I_2,$$

for $I_1 \neq I_2$, where $\{\zeta(t), t \in T\}$ is a classical stochastic process defined as $\zeta(t) = A\cos(t) + B\sin(t)$ for A and B being random variables defined as

	A	B
Prob.	$\frac{1}{3}$	$\frac{2}{3}$

Now, $\mu_{\mathcal{N}_r}(t) = 0$ and
$\mathcal{R}_{\mathcal{N}_r}(s,t) = 2\cos(\tau) + 6\cos(\tau)I_1 + 6\cos(\tau)I_2 = \mathcal{R}_{\mathcal{N}_r}(\tau)$.
Therefore, this shows that $\{\mathcal{N}_r(t), t \in T\}$ is weakly stationary.

If $I_1 = I_2$, $\mathcal{N}_r(t)$ becomes a classical neutrosophic stochastic process and the condition satisfies as well.

3 Conclusion

In this paper we have defined the notion of refined neutrosophic stochastic process, which can be represented in $\mathbb{R}^n$ for n-classical stochastic processes. The first one is $\{\zeta(t), t \in T\}$ and the rest are $\left\{\zeta(t) + \sum_{i=1}^{n}\eta_n, t \in T)\right\}$, where the dependence is on the number of indeterminacies that the problem has. Many results were obtained in the classical way and by using n-dimensional AH-isometry.

The results obtained in this paper can be applied in several real situations in different fields of mathematics and other sciences such as finance, biology, decision-making, and so on. Besides, we encourage the reader to explore potential applications of this concept.

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5 Conflicts of Interest

The authors declare no conflict of interest.

6 Data availability statement

This manuscript has no associated data.

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