Vol. 15 (núm 1) pp. 150 - 154
Enero - Junio 2024
https://doi.org/10.19053/01217488.v15.n1.2024.16629

Aplicación de un método de cociente de diferencias de dos lados en la solución de un problema inverso mal puesto no lineal de una ecuación elíptica auto-adjunta

Application of the Two-sided difference quotient in the solution of nonlinear Ill-posed inverse self-adjoint elliptic problem

Luis Eduardo Olivar Robayo, [1,*], Héctor Andrés Granada Díaz,[1,**]

Abstract

When we use a discretization by finite differences, to solve differential equations we find problems at the border of the domain of the solution. If the solution is also immersed in a ill-posed inverse problem; we can find very bad solutions. In this paper we apply a discretization of two - sided difference quotients method to solve Ill-posed inverse self-adjoint elliptic problem [1]. Some numerical examples showing the effectiveness of this method and we will use mollification techniques to smooth the solutions.

Keywords: Inverse problems, regularization methods, elliptic equations, ill-posed problems, mollification methods

Resumen

Cuando utilizamos una discretización por diferencias finitas para solucionar ecuaciones diferenciales, encontramos problemas en la frontera del dominio de la solución; si además la solución esta inmersa en un problema inverso mal puesto, podemos encontrar soluciones muy malas. En este artículo aplicamos una discretización del cociente de diferencias de dos lados para resolver un problema elíptico autoadjunto inverso mal puesto [1]. Mostraremos algunos ejemplos numéricos que muestran la efectividad de este método y usaremos técnicas de molificación para suavizar las soluciones

Palabras Clave: Problemas inversos, métodos de regularización, ecuaciones elípticas, problemas mal puestos, métodos de molificación

Recepción: 04-Octubre-2023
Aceptación: 15-Diciembre-2023

[1*] Departamento de Matemáticas y estadística, Grupo de Matemáticas del Tolima (Grupo-MaT), Universidad del Tolima. Dirección electrónica: leolivart@ut.edu.co
[1**] Departamento de Matemáticas y estadística, Grupo de Matemáticas del Tolima (Grupo-MaT), Universidad del TolimaḊirección electrónica: hagranadad@ut.edu.co

1 Introduction

Identification problems in some basic differential equations play an important role in many branches of science and engineering. Several discretizations are reported to numerically solve the elliptic problem. However, when it comes to solving an inverse problem such as finding a coefficient, numerical approximation schemes can give undesired results and even more so when the information presented is contaminated with noise. The discretization of the problem is done using two difference quotients which is a regularization strategy (see [1]) can solve the problem and molification smooth the desired result. Hinestroza and Murio introduced a modification method for the identification of coefficients (see [2]). In this paper we consider the inverse problem of identifying the coefficient: find the thermal conductivity $a(x) > 0$ in the parabolic equation:

$$\begin{cases} u_t & = & (a(x)u_x(x,t))_x + g(x,t), \\ & & (x,t) \in [0,1] \times [0,1], \\ u(x,0) & = & \varphi(x), \\ u(1,t) & = & 0, \\ u(0,t) & = & 0. \end{cases} \tag{1}$$

in the (1) equation, $g(x,t)$ and $u(x,t)$ and $\varphi(x)$ are given or partly known, the subscript x denotes derivative with respect to the variable x. In addition, $a(x) > 0$.

By these conditions, we will denote

$$u(x) = \int_0^1 u(x,t)dt, \qquad h(x) = \int_0^1 g(x,t)dt,$$

$$f(x) = h(x) - \varphi(x) + u(x,1).$$

Then, we reduce (1) to the elliptic equation as a inverse steady heat conduction problem (2).

$$\begin{cases} (a(x)u_x(x))_x & = f(x),\ x \in [0,1], \\ u(1) & = 0, \\ u(0) & = 0. \end{cases} \tag{2}$$

If u_x does not vanish anywhere in (2) and $a(0) = 0$, we can identify a explicitly as

$$a(x) = \frac{\int_0^x f(s)ds}{u_x} \tag{3}$$

Therefore, it is determined uniquely. So, in order to calculate a, one has to differentiate u, which is an Ill-posed problem (see [1]).

Furthermore, there is another effect of instability from the division by u_x in (3): in regions where u_x is small, errors, for example, f can occur, which is not surprising since when u_x vanishes, a cannot be determined at all, so some instability must be expected where u_x is small. This is a nonlinear effect, while pool conditions involved with differentiation data derived from the fact that the linearized problem is too bad. We emphasize that, in general, the parameter identification is a nonlinear inverse problem, even if the underlying equation is (for a known parameter) a linear equation. This, coupled with inverse problems in heat conduction and inverse scattering problem (see [3]), are strong motivations for the study of nonlinear inverse problems.

This paper is organized as follows: In section 2 we present the Two different quotients method. In section 3 we will show the modification method for the identification of coefficients that would allow smoothing the estimate of the coefficient and show examples of parameter estimation under conventional discretization methods and will be compared with the double quotient discretization method. Then, the estimate will be smoothed by mollification method.

2 The two different quotients discretization method

To reconstruct a in (2), we take into account temperature measurements.

$$\begin{aligned} u_i &= u(x_i), \quad u^{\delta}(x_i) = u_{i,} + E(\delta) \\ f_i &= f(x_i), \quad f^{\delta}(x_i) = f_{i,} + E(\delta). \end{aligned} \tag{4}$$

where $i = 1, \cdots, n$ and $E(\delta)$ is noise, which depends of δ.

The discretization of the problem is done (2) using two difference quotients which is a regularization strategy $(R_h y)(x)$ (see [1]).

$$\begin{aligned} &\frac{1}{h}\left[4y\left(x+\frac{h}{2}\right) - y(x+h) - 3y(x)\right], 0 < x < \frac{h}{2}, \\ &\frac{1}{h}\left[y\left(x+\frac{h}{2}\right) - y\left(x-\frac{h}{2}\right)\right], \frac{h}{2} < x < 1-\frac{h}{2} \\ &\frac{1}{h}\left[3y(x) + y(x-h) - 4y\left(x-\frac{h}{2}\right)\right], 1-\frac{h}{2} < x < 1. \end{aligned} \tag{5}$$

Where regularization parameter $\alpha = h$ is the step size in (5). Having the problem written in this form

$$A(u)a = f, A \in \mathbb{R}^{n\times n}, f \in \mathbb{R}^n. \tag{6}$$

Where $A(u)$ is non singular. The discretization equation (2) can be taken to the form f_i for $i = 1, \cdots, n$ by equation (7).

$$
\begin{aligned}
f_1 = & \frac{1}{h^2}\left[(u_{i+1}-u_{i+2})a_i\right.\\
& \left.+(5u_{i+1}-2u_{i+2}-3u_i)a_{i+1}\right],\\
f_{2\leq i\leq n-1} = & \frac{1}{2h^2}\left[(u_{i-1}-u_i)a_{i-1}\right.\\
& \left.+(u_{i+1}-2u_i+u_{i-1})a_i+(u_{i+1}-u_i)a_{i+1}\right],\\
f_n = & \frac{1}{h^2}\left[(5u_{i-1}-2u_{i-2}-3u_i)a_{i-1}\right.\\
& \left.+(u_{i-1}-u_{i-2})a_i\right].
\end{aligned}
\tag{7}
$$

3 Numerical calculations and examples

For the computations of examples 1 and 2 we use MATLAB R2021a. In subsection 3.1 the methodology to introduce the noise of the system $Aa = f^{\delta}$ is presented, in section 3.2 the procedure to smooth the solution.

3.1 Noise generation for the system $A(u)a = f$

For the $Aa = f$ matrix system obtained under discretization methods (7) that follows approximate the estimate of the functional $a(x) > 0$ we introduce a noise of the order of 10^{-2}:

$$
f^{\delta}(x) = f(x) + E(\delta), \tag{8}
$$

where E is a noise vector whose entries are chosen from a normal distribution with mean 0 and variance 1, so that:

$$
\frac{\|E(\delta)\|}{\|f_i\|} = \frac{\|f_i - f_i^{\delta}\|}{\|f_i\|} \leq \delta = 0.01 \tag{9}
$$

3.2 Smooth by mollification

We use the Gaussian Kernel Ψ_h by

$$
\Psi_h(x) = \frac{1}{h\sqrt{\pi}}\exp\left(-x^2/h^2\right) \tag{10}
$$

and the convolution

$$
\begin{aligned}
(\Psi_h * a)(x) &= \int_{-\infty}^{\infty} \Psi_h(x-s)a(s)ds\\
&= \int_{-\infty}^{\infty} \Psi_h(s)a(x-s)ds, \quad x \in \mathbb{R}.
\end{aligned}
\tag{11}
$$

For more information on the mollification method see [1].

3.3 Numerical examples

In the following examples, $f(x)$, $u(x)$ and $a(x)$ are given to satisfy problem (2). Therefore, we will denote the exact solution as a, a_T^{δ} is the approximation of a by using trapezoidal-rule quadrature with $n-1$ trapezoids that approximate the integral given in equation (3) whose integrand is perturbed as shown in equation (8) to obtain the following equation:

$$
a_T^{\delta}(x) = \frac{\int_0^x f^{\delta}(s)ds}{u_x(x)}. \tag{12}
$$

Given $n = 64$ data that defines the problem, calculate the step size $h = 1/(n-1)$. We denote by a_R^{δ} the solution of system $A(u)a_R^{\delta} = f^{\delta}$ which originates through the discretization of equation (7). Finally, we denote by a_M^{δ} the smoothing of the approximation solution a_R^{δ} by Mollification, see section 3.2. We also note by E_r as the relative error of approximating f with f^d, E_{rm} as the relative error of approximating a with a_M^{δ} and the error E_M as error by molification defined as:

$$
\begin{aligned}
E_r &= \frac{\|E(\delta)\|}{\|f_i\|}, \quad E_{rm} = \frac{\|a_M^{\delta} - a\|}{\|a\|},\\
E_M &= \sqrt{\sum_{k=1}^{n}\frac{1}{2n-1}\left(a_M^{\delta}(k) - a(kh)\right)^2}.
\end{aligned}
$$

exmp 1 *Consider the functions f, u and a given in the numerical example 1 of ([5]).*

$$
\begin{aligned}
f(x) = & \frac{40(1-40x^2)}{(40x^2+1)^2}\left(\sin(\pi x) + \pi\cos(\pi x)\right)e^x\\
& + \frac{40x^2+40x+1}{40x^2+1}\left((1-\pi^2)\sin(\pi x)\right.\\
& \left.+2\pi\cos(\pi x)\right)e^x,
\end{aligned}
\tag{13}
$$

$$
u(x) = e^x\sin(\pi x), \quad a(x) = \frac{40x}{40x^2+1} + 1. \tag{14}
$$

Figure 1 shows the graphs of f and f^{δ}, where it can be seen that the induced noise is less than 0.01 ($E_r = 0.01$), zooming allows you to see the difference between the two functions.

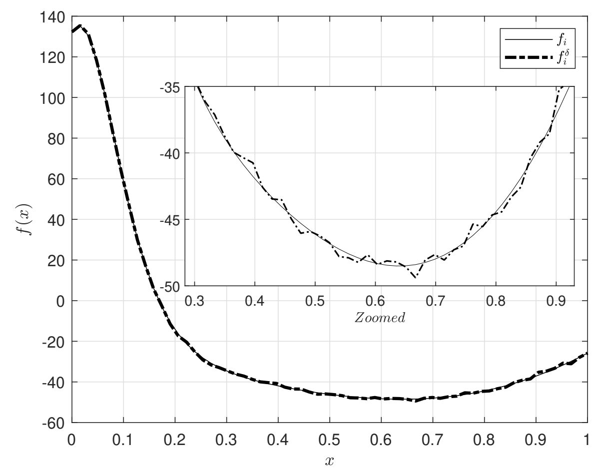

Figure 1: Numerical f and f^{δ}. Example 1.

Figure 2 shows Exact Solution a which is continuous for all $x \in [0,1]$ and numerical solution without regularizing under the trapezoidal method a_T^{δ} that presents an asymptotic behavior at $x = 0.598$ when u_x vanishes, see equation (12).

Additionally, its approximation in its entire domain is not good and for some values of x, the function $a(x) < 0$.

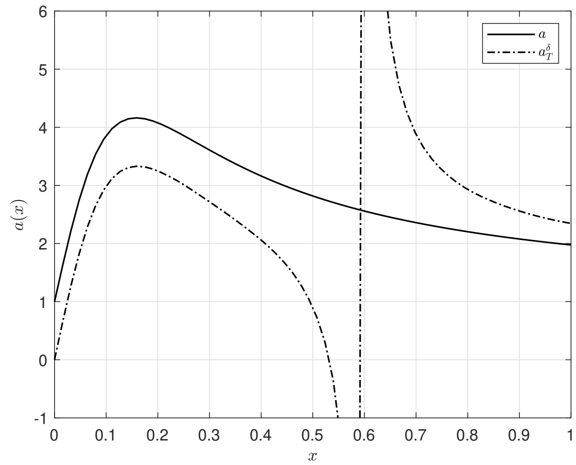

Figure 2: Exact Solution a and numerical solution without regularizing under the trapezoidal method a_T^δ. Example 1.

Figure 3 shows that the discretization of equation (7) *regularizes the problem and allows a good approximation of a, note that the approximation does not have an asymptotic behavior.*

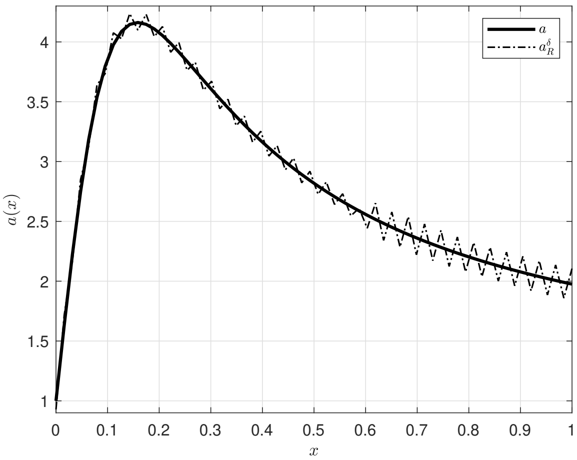

Figure 3: Exact Solution and numerical regularized solution a_R^δ. Example 1.

Figure 4 shows a_M^δ*, the smoothing of the approximation solution* a_R^δ *by Mollification. where it is observed that the solution* a_R^δ *shown in Figure 3 is smoothed, and a good approximation to the exact solution is obtained as indicated by the respective errors* $E_{rm} = 0.0124$ *and* $E_M = 0.12825$.

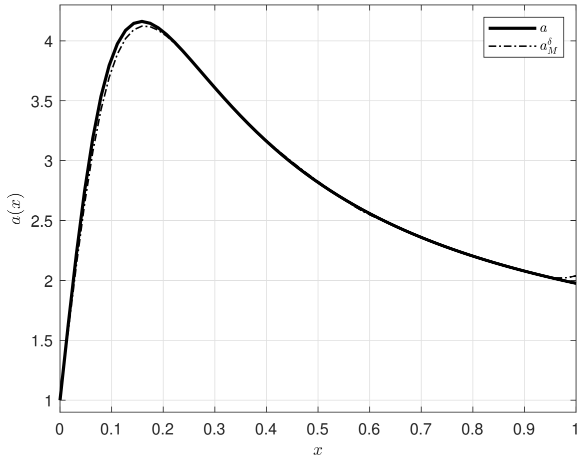

Figure 4: Exact Solution and numerical regularized solution a_R^δ by Mollification. Example 1.

exmp 2 *Consider the functions f,u and a given in the numerical example 1 of ([4]).*

$$f(x) = \frac{20e^x(\sin(\pi x) + \pi\cos(\pi x))}{(40x-24)^2+1} + \frac{(\tan^{-1}(40x-24)+3)e^x}{2}\big(\sin(\pi x) + 2\pi\cos(\pi x) - \pi^2\sin(\pi x)\big) \tag{15}$$

$$u(x) = e^x\sin(\pi x), \quad a(x) = 3/2 + (1/2)\tan^{-1}(40(x-3/5)). \tag{16}$$

Figure 5 shows graphs of f and f^δ*, in this example* $E_r = 0.0097$.

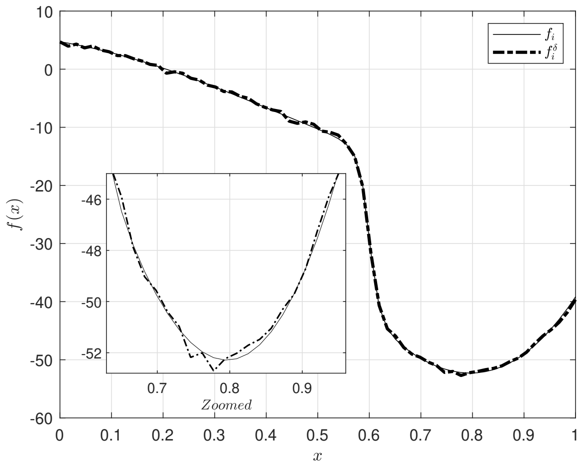

Figure 5: Numerical f and f^δ. Example 2.

In Figure 6 you can see again the asymptotic behavior where by trapezoidal method make that for some values of x the *function a is negative. It can also be seen that the regularization method allowed us to approximate the functional a.*

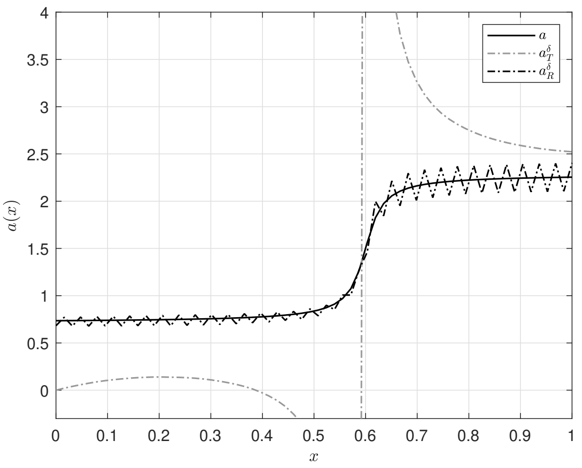

Figure 6: Exact Solution, a and numerical solution without regularizing under the trapezoidal method a_T^δ and numerical regularized solution a_R^δ. Example 2.

Finally, in Figure 7 it is smoothed by mollification to the regularized solution and the errors $E_{rm} = 0.0419$ *and* $E_M = 0.0583$ *were obtained.*

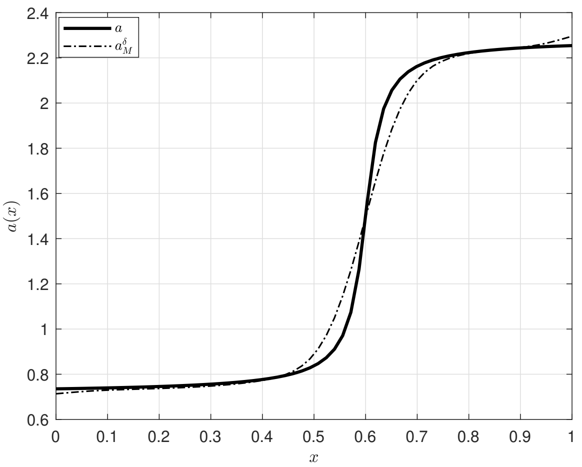

Figure 7: Exact Solution and numerical regularized solution a_R^δ by Mollification. Example 2.

4 Conclusions

The derivative regularization method proposed by Kirch applied to self-adjoint elliptic equations to solve the ill posed inverse problem of finding the conductivity coefficient; it is easy to program and shows very good results compared to the proposed methods and examples. In the examples, mollification was used to smooth the graphs and not as a regularization method. [4, 5]. It would be interesting for the future, a comparison with other methods or at least show the complexity of the proposed method compared to traditional ones. Although due to space limitations it would be interesting to see the convergence.

Author Contributions:

Conceptualization, investigation, methodology, software, formal analysis, writing - review, and editing: L.E.Olivar R. and H.A. Granada D. All authors have read and agreed to the published version of the manuscript.

Funding:

This research received no external funding.

Acknowledgments:

The work of H. A. Granada D. and L. E. Olivar R. was supported by Universidad del Tolima, Ibagué-Colombia.

Conflicts of Interest:

The authors declare no conflict of interest.

References

[1] Kirsch, Andreas, *An introduction to the mathematical theory of inverse problems*,Springer: New York, NY [u.a.] 2011.

[2] Hinestroza, Doris and Murio, Diego A., Identification of transmissivity coefficients by mollification techniques. Part I: one-dimensional elliptic and parabolic problems. *Elsevier: Computers & Mathematics with Applications* **1993**, *25(8)*, 59–79.

[3] Engl, Heinz Werner and Hanke, Martin and Neubauer, Andreas., *Regularization of inverse problems*; Springer: Science & Business Media 1996.

[4] Hinestroza G., Doris and Murio, Diego A. and Zhan, S., Regularization techniques for nonlinear problems. *Computers &; Mathematics with Applications* **1999**, *37(10)*, 145–159.

[5] Hinestroza G., Doris and Peralta, Jenifer and Olivar, Luis Eduardo, Regularization algorithm within two parameters for the identification of the heat conduction coefficient in the parabolic equation. *Mathematical and Computer Modelling* **2013**, *57*, 1990–1998.