Revista Ciencia en Desarrollo, Vol. 6 No. 2
ISSN 0121-7488 – Julio-Diciembre de 2015, pp. 205-213

Some Relations between N-Koszul, Artin-Schelter Regular and Calabi-Yau Algebras with Skew PBW Extensions

Algunas relaciones entre álgebras N-Koszul, Artin-Schelter regular y Calabi-Yau con extensiones PBW torcidas

H. Suárez[a,*]
O. Lezama[b]
A. Reyes[b]

Recepción: 05-ene-15
Aceptación: 15-jun-15

Abstract

Some authors have studied relations between Artin-Schelter regular algebras, N-Koszul algebras and Calabi-Yau algebras (resp. skew Calabi-Yau) of dimension d. In this paper we want to show through examples and counterexamples some relations between these classes of algebras with skew PBW extensions. In addition, we also exhibit some examples of the preservation of these properties by Ore extensions.

***Key words*:** Skew PBW extensions, Calabi-Yau algebras, N-Koszul algebras, AS-regular algebras, Ore extensions.

Resumen

Algunos autores han estudiado las relaciones entre las álgebras Artin-Schelter regular, las álgebras N -Koszul y las álgebras Calabi-Yau (resp. skew Calabi-Yau) de dimensión d . En este artículo queremos mostrar a través de ejemplos y contraejemplos algunos relaciones entre estas clases de álgebras y las extensiones PBW torcidas. Además, mostraremos algunos ejemplos de preservación de estas propiedades en las extensiones de Ore.

***Palabras clave*:** Skew PBW extensions, Calabi-Yau algebras, N-Koszul algebras, AS-regular algebras, Ore extensions.

[a]Escuela de Matemáticas y Estadística, Universidad Pedagógica y Tecnológica de Colombia, Tunja.
[*]Autor de correspondencia: hector.suarez@uptc.edu.co
[b]Seminario de Álgebra Constructiva - SAC^2, Departamento de Matemáticas, Universidad Nacional de Colombia, sede Bogotá.

1. Introduction

Recently there have been defined some special classes of algebras such as N-Koszul algebras, Calabi-Yau algebras and skew *PBW* extensions. Koszul algebras, which in this article are called 2-Koszul algebras were introduced by Stewart B. Priddy in [34]. Later in 2001, Roland Berger in [3] introduces a generalization of Kozsul algebras, which are then called generalized Koszul algebras or N-Koszul algebras. In [17] Victor Ginzburg defined d-Calabi-Yau algebras or Calabi- Yau algebras of dimension d (or simply Calabi-Yau algebras). Then in [6], Roland Berger and Rachel Taillefer introduced the definition of graded Calabi-Yau algebra. As a generalization of Calabi-Yau algebras, were also defined the skew Calabi-Yau algebras. On the other hand, the skew *PBW* extensions were introduced in 2011 by Oswaldo Lezama and Claudia Gallego in [16].

In the current literature, it has been studied certain relations between Artin Schelter regular algebras, N-Koszul algebras, Calabi-Yau algebras and skew Calabi-Yau algebras. Our aim is to show through a serie of examples some relationships between the above algebras and skew *PBW* extensions. Unless otherwise specified, throughout this article, $\mathbb{K}$ will represent a fixed but arbitrary field.

2. Definitions and Elementary Properties

2.1. *AS*-Regular Algebras

Regular algebras were defined by Michael Artin and William Schelter in [2]. They studied the regular algebras of global dimension three which are generated by elements of degree one and classified into thirteen types.

Definition 1 ([2])**.** Let $A = \mathbb{K}\oplus A_1 \oplus A_2 \oplus \cdots$ be a finitely presented graded algebra over $\mathbb{K}$. The algebra A will be called **regular** if it has the following properties:

(i) A has finite global dimension d: every graded $A-$module has projective dimension $\leq d$.

(ii) A has finite Gelfand-Kirillov dimension (GKdim), i.e., A has polynomial growth.

(iii) A is *Gorenstein*,i.e., $Ext^q_A(\mathbb{K}, A) = 0$ if $q \neq d$, and $Ext^d_A(\mathbb{K}, A) \cong \mathbb{K}$.

In the current literature these algebras are called Artin-Schlter regular algebras (*AS*-regular algebras). Most of the authors do not consider the condition (*ii*) in the definition of *AS*-regular algebras. We say that A has *polynomial growth* if there exist $c \in \mathbb{R}^+$ and $r \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $dim_{\mathbb{K}} A_n \leq cn^r$.

2.2. N-Koszul Algebras

Koszul algebras were defined by Stewart B. Priddy in [34], later in 2001, Roland Berger in [3] introduces a generalization of Koszul algebras which are called *generalized Koszul algebras or N-Koszul algebras*. Koszul algebras defined by Stewart B. Priddy correspond to 2-Koszul algebras in this paper.

Definition 2 ([3])**.** The **generalized Koszul algebras** are graded algebras $A = \mathbb{K}\oplus A_1 \oplus A_2 \oplus \cdots$ which are generated in degrees 0 and 1 such that there is a graded projective resolution of $\mathbb{K}$

$$\cdots \to P^i \to P^{i-1} \to \cdots \to P^0 \to \mathbb{K} \to 0,$$

such that for any $i \geq 0$, P^i is generated in degree $\delta(i)$, where

$$\delta(i) = \begin{cases} \frac{i}{2}N, & \text{if } i \text{ is even;} \\ \frac{i-1}{2}N + 1, & \text{if } i \text{ is odd,} \end{cases}$$

for some $N \geq 2$.

If $N = 2$, N-Koszul algebras is usually called Koszul. In this situation, Definition 2 coincides with that given by Stewart Priddy in [34].

2.3. Calabi-Yau Algebras of Dimension d

Calabi-Yau algebras of dimension d or d-Calabi-Yau algebras were defined by Victor Ginzburg in [17].

Definition 3 ([17], Definition 3.2.4)**.** A $\mathbb{K}$-algebra A is called a **Calabi-Yau algebra of dimension** d if

(i) A is homologically smooth; that is, A has a finite resolution of finitely generated projective A-bimodules;

(ii) $Ext^i_{A-Bim}(A, A \otimes A) \cong \begin{cases} A, & \text{if } i = d \\ 0, & \text{if } i \neq d, \end{cases}$
as A-bimodules.

The space $A \otimes A$ is endowed with two A-bimodule structures: the outer structure defined by $a \cdot (x \otimes y) \cdot b = ax \otimes yb$, and the inner structure defined by

$a \cdot (x \otimes y) \cdot b = xb \otimes ay$. Consequently, the Hom spaces $Hom_{A-A}(M, A \otimes A)$ of A-bimodule morphisms from M to $A \otimes A$ endowed with the outer structure are again A-bimodules using the inner structure of $A \otimes A$, and the same is true for the Hochschild cohomology spaces $H^k(A, A \otimes A)$. For $A^e = A \otimes A^{op}$, the enveloping algebra of A, each A-bimodule M is a left A^e-module for the action $(a \otimes b).m = amb$ and right A^e-module for the action $m.(a \otimes b) = bma$.

Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be a $\mathbb{Z}$-graded algebra, and $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a graded A-bimodule. For any integer l, $M(l)$ is a graded A-bimodule whose degree i component is $M(l)_i = M_{i+l}$.

Definition 4. A graded algebra A is called a **graded Calabi-Yau algebra of dimension** d if

(i) A has a finite resolution of finitely generated graded projective A-bimodules, and

(ii) $Ext^i_{A^e}(A, A \otimes A) \cong \begin{cases} 0, & \text{if } i \neq d \\ A(l), & \text{if } i = d, \end{cases}$
as graded A-bimodules; for some integer l.

It follows from Definition 4 that every graded Calabi-Yau algebra of dimension d is Calabi-Yau of dimension d (see [6], Proposition 4.3).

Let M be an A-bimodule, $\nu, \mu : A \to A$ two automorphism, the ***skew*** A-bimodule ${}^{\nu}M^{\mu}$ is equal to M as a vector $\mathbb{K}$-space whit $a \cdot m \cdot b = \nu(a)m\mu(b)$.

Definition 5. Let A be a $\mathbb{K}-$algebra. A is called **skew Calabi-Yau** of dimension d if there exists an automorphism ν of A such that

(i) A is homologically smooth; and

(ii) $Ext^i_{A^e}(A, A^e) \cong 0$ when $i \neq d$ and $Ext^d_{A^e}(A, A^e) \cong {}^1A^{\nu}$ as A^e-modules.

In this case, ν is called the ***Nakayama Automorphism*** of A. The Nakayama automorphism is unique up to an inner automorphism. A ν-skew Calabi-Yau algebra A is Calabi-Yau in the sense of Ginzburg if and only if ν is an inner automorphism of A (see [30], Definition 1.1). So every Calabi-Yau algebra is skew Calabi-Yau.

2.4. Skew *PBW* Extensions

Skew PBW extensions or $\sigma - PBW$ extensions were defined in 2011 by Oswaldo Lezama and Claudia Gallego in [16].

Definition 6. Let R and A be rings. We say that A is a **skew** PBW **extension** of R if the following conditions hold:

(i) $R \subseteq A$.

(ii) There exist elements $x_1, \ldots, x_n$ in A such that A is a left free R-module, with basis,
$$Mon(A) := \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}.$$

(iii) For each $1 \leq i \leq n$ and any $r \in R - \{0\}$ there exists an element $c_{i,r} \in R - \{0\}$ such that
$$x_i r - c_{i,r} x_i \in R.$$

(iv) For any elements $1 \leq i, j \leq n$, there exists $c_{i,j} \in R - \{0\}$ such that
$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n.$$

Proposition 1 ([16], Proposition 3)**.** *Let A be a skew PBW extension of R. Then, for every $1 \leq i \leq n$, there exist an injective ring endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that*
$$x_i r = \sigma_i(r)x_i + \delta_i(r),$$
for each $r \in R$.

In this case we write $A := \sigma(R)\langle x_1, \ldots, x_n\rangle$.

We say that A is a ***bijective*** if σ_i is bijective for each $1 \leq i \leq n$ and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$ (see [16], Definition 4).

3. Relations, Examples and Counterexamples

Some authors have found some interesting relations between AS-regular algebras, N-Koszul algebras and Calabi-Yau algebras. Some examples of these relations are the following:

(i) Roland Berger and Nicolas Marconnet in Proposition 5.2 of [8] show that if $A = T(V)/\langle R\rangle$ is a connected graded $\mathbb{K}$-algebra such that the space V of generators is concentrated in degree 1, the space R of relations lives in degrees ≥ 2, the global dimension d of A is 2 or 3, and that A is AS-regular (the polynomial growth imposed by Artin and Schelter is often removed and in fact, it is not necessary), then A is N- Koszul if $d = 3$, and 2-Kozul if $d = 2$.

(ii) Roland Berger y Rachel Taillefer in Proposition 4.3 of [6] show than if A is a connected $\mathbb{N}$-graded

Calabi-Yau algebra then A is AS-regular algebra, and in Proposition 5.4 they prove that if A is AS-regular $\mathbb{C}$-algebra of global dimension 3 (with polynomial growth), then A is Calabi-Yau if and only if A is of type $\mathtt{A}$ in the classification of Artin and Schelter given in [2].

(iii) Let $\mathbb{K}$ be of characteristic zero, V be an n-dimensional space with $n \geq 1$, w be a non-zero homogeneous potential of V of degree $N+1$ with $N \geq 2$, and $A = A(w)$ be the potential algebra defined by w (so that the space of generators of A is V); Roland Berger and Andrea Solotar in Theorem 2.6 of [4] prove that if the space of relations R (i.e. the subspace of $V^{\otimes N}$ generated by the relations $\partial_x(w)$, $x \in X$) of A is n-dimensional, then A is 3-Calabi-Yau if and only if A is N-Koszul of global dimension 3 and $dim R_{N+1} = 1$, where $R_{N+1} = (R \otimes V) \cap (V \otimes R) \subseteq V^{\otimes(N+1)}$.

(iv) Manuel Reyes, Daniel Rogalski and James Zhang in Lemma 1.2 of [37] show that if A is a connected graded algebra, then A is graded skew Calabi-Yau if and only if A is AS-regular.

3.1. Examples

In the current literature there are not explicit relations between skew PBW extensions with AS-regular algebras, N-Koszul algebras or Calabi-Yau algebras. Next we will show some examples of algebras that are AS-regular, or N-Koszul, or Calabi-Yau, or a combination of these types, that are skew PBW extensions.

3.1.1. AS-regular + N-Koszul + Calabi-Yau

Below are some examples of algebras that are AS-regular, N-Koszul and Calabi-Yau, and in addition, they are also skew PBW extensions.

1. The polynomial algebra $A = \mathbb{K}[x, y]$ is a connected graded Noetherian algebra of global dimension 2. It follows that A is AS-regular with $GKdim(A) = 2$ (see [40], Theorem 3.5), A is 2-Koszul algebra (see [8], Proposition 5.2). Moreover, A is Calabi-Yau of dimension 2 (see [28]), and A is a skew PBW extension (see [16], Example 5).
2. Let $A = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial algebra in n variables. Then A is a 2-Kozsul algebra (see [31], Example 1.6), A is a skew PBW extension (see [16], Example 5), A is Calabi-Yau of dimension n (see [9], page 18) and therefore, AS-regular (see [6], Proposition 4.3).
3. Let $A = \mathbb{K}\langle x, y, z\rangle / \langle yz - zy, zx - xz, xy - yx + z^2\rangle$ which is of type S'_1 in the classification of three-dimensional AS-regular algebras given in [2]. According to [8], A is 3−Calabi-Yau (see [45], Example 3.6), and by Proposition 5.2 of [8] A is 2-Koszul. We note that $A \cong \sigma(\mathbb{K}[z])\langle x, y\rangle$ and therefore A is a skew PBW extension.
4. For any $n \geq 2$, let A be a non-degenerate noncommutative quadric graded algebra in n variables $x_1, \ldots, x_n$ of degree 1. Let z be an extra variable of degree 1. Let B be an algebra defined by a non-zero cubic potential w in the variables $x_1, \ldots, x_n$, z. Assume that the graded algebra B is isomorphic to a skew polynomial algebra $A[z; \sigma; \delta]$ over A in the variable z, defined by a 0-degree homogeneous automorphism σ of A and a 1-degree homogeneous σ−derivation δ of A. Then B is 2-Koszul and 3-Calabi-Yau (see [4], Proposition 4.1). B is a skew PBW extension.

3.1.2. AS-Regular + N-Koszul

The following are some examples of AS-regular N-Koszul algebras which are skew PBW extensions. It is not clear if these algebras are Calabi-Yau or not, since we have no clear criteria for making claims in this regard.

1. The algebra $A = \mathbb{K}\langle x, y, z\rangle / \langle \alpha\beta xy + a\alpha\beta yx, \alpha zx + axz, yz + a\beta zy\rangle$ is AS-regular of global dimension 3 of type S_1 (see [2], Theorem 3.10). Moreover, A is 2-Koszul (see [8], Proposition 5.2), and A is a skew PBW extension.

   A may be or not Calabi-Yau, depends on the coefficients a, α and β (see [6], Proposition 5.4).
2. The quantum plane $A = \mathbb{K}\langle x, y\rangle / \langle yx - cxy\rangle$ ($c \neq 0$) is an AS-regular algebra of global dimension 2 (see [2], page 172), Moreover A is a skew PBW extension as well as 2-Koszul (see [8], Proposition 5.2). For example, if $c = 1$ then the quantum plane A is a 2-Calabi-Yau algebra.
3. The Jordan plane $A = \mathbb{K}\langle x, y\rangle / \langle yx - xy - x^2\rangle$ is an AS-regular algebra of global dimension 2 (see [2], page 172). Since A is a quadratic algebra and $\langle yx - xy - x^2\rangle$ is a principal ideal, it

follows that A is 2-Koszul (see [15], page 7), $A \cong \sigma(\mathbb{K}[x])\langle y\rangle$ and therefore A is a skew *PBW* extension. The Jordan plane A is not Calabi-Yau (see [30]).

3.1.3. *Skew Calabi-Yau algebras*

The following is an example of skew Calabi-Yau algebra that is skew *PBW* extension. Multi-parameter quantum affine $n-$spaces $O_{\mathbf{q}}(\mathbb{K}^n)$ can be obtained by iterated Ore extensions. Let $n \geq 1$ and $\mathbf{q}$ be a matrix $(q_{ij})_{n\times n}$ whit entries in a field $\mathbb{K}$ where $q_{ii} = 1$ y $q_{ij}q_{ji} = 1$ for all $1 \leq i, j \leq n$. Then quantum affine n-space $O_{\mathbf{q}}(\mathbb{K}^n)$ is defined to be $\mathbb{K}-$algebra generated by $x_1, \cdots, x_n$ with the relations $x_jx_i = q_{ij}x_ix_j$ for all $1 \leq i, j \leq n$. The $\mathbb{K}-$algebra $O_{\mathbf{q}}(\mathbb{K}^n)$ is skew Calabi-Yau whit the Nakayama automorphism ν such that $\nu(x_i) = (\prod_{j=1}^{n} q_{ji})x_i$ (see [30], Proposition 4.1). This $\mathbb{K}-$algebra is a skew *PBW* extension (see [29]).

The Jordan plane $A = \mathbb{K}\langle x, y\rangle/\langle yx - xy - x^2\rangle$ is skew Calabi-Yau, but not Calabi-Yau (see [30]).

3.1.4. *The universal enveloping algebra and the Sridharan enveloping algebra of Lie algebra*

Let $\mathcal{G}$ be a finite dimensional Lie algebra over $\mathbb{K}$ with basis $\{x_1, \cdots, x_n\}$. The universal enveloping algebra of $\mathcal{G}$, denoted $\mathcal{U}(\mathcal{G})$, is a *PBW* extension of $\mathbb{K}$ since $x_ir - rx_i = 0$, $x_ix_j - x_jx_i = [x_i, x_j] \in \mathcal{G} = \mathbb{K} + \mathbb{K}x_1 + \cdots + \mathbb{K}x_n$, $r_i \in \mathbb{K}$, for $1 \leq i, j \leq n$. Ji-Wei He, Fred Van Oystaeyen and Yinhuo Zhang showed that for the 3-dimensional Lie algebra $\mathcal{G}$ with basis $\{x, y, z\}$, $\mathcal{U}(\mathcal{G})$ is a Calabi-Yau algebra if and only if the Lie bracket is given by $[x, y] = ax + by + wz$, $[x, z] = cx + vy - bz$, $[y, z] = ux - cy + az$, where $a, b, c, u, v, w \in \mathbb{K}$; and if $\mathcal{G}$ is a finite dimensional Lie algebra, $\mathcal{U}(\mathcal{G})$ is Calabi-Yau of dimension 3 if and only if $\mathcal{G}$ is isomorphic to one of the following Lie algebras (see [22], Proposition 4.5 and Proposition 4.6):

(i) The 3$-$dimensional simple Lie algebra $sl(2, \mathbb{K})$;

(ii) $\mathcal{G}$ has a basis $\{x, y, z\}$ such that $[x, y] = y$, $[x, z] = -z$ and $[y, z] = 0$;

(iii) The Heisenberg algebra, that is; $\mathcal{G}$ has a basis $\{x, y, z\}$ such that $[x, y] = z$ and $[x, z] = [y, z] = 0$;

(vi) The 3$-$dimensional abelian Lie algebra.

We note that if $\mathcal{G}$ is a finite dimensional Lie algebra over a field $\mathbb{K}$ and $\mathcal{U}(\mathcal{G})$ is the universal enveloping algebra of $\mathcal{G}$, then $\mathcal{U}(\mathcal{G})$ is a skew *PBW* extension (see [16]); in particular, universal enveloping Calabi-Yau algebra $\mathcal{U}(\mathcal{G})$ of dimension 3 is a skew *PBW* extension.

Let $\mathcal{G}$ be a finite dimensional Lie algebra, and let $f \in Z^2(\mathcal{G}, \mathbb{K})$ be an arbitrary 2$-$***cocycle***, that is, $f : \mathcal{G}\times\mathcal{G} \to \mathbb{K}$ such that $f(x, x) = 0$ and

$$f(x, [y, z]) + f(z, [x, y]) + f(y, [z, x]) = 0$$

for all $x, y, z \in \mathcal{G}$.

The ***Sridharan enveloping*** algebra of $\mathcal{G}$ is defined to be the associative algebra $\mathcal{U}_f(\mathcal{G}) = T(\mathcal{G})/I$, where I is the two-side ideal of $T(\mathcal{G})$ generated by the elements

$$(x\otimes y) - (y\otimes x) - [x, y] - f(x, y), \text{ for all } x, y \in \mathcal{G}.$$

For $x \in \mathcal{G}$, we still denote by x its image in $\mathcal{U}_f(\mathcal{G})$. $\mathcal{U}_f(\mathcal{G})$ is a filtered algebra with the associated graded algebra $gr(\mathcal{U}_f(\mathcal{G}))$ being a polynomial algebra.

Let $\mathbb{K}$ be a field and algebraically closed whit characteristic zero. If $\mathcal{G}$ is a Lie $\mathbb{K}$-algebra of dimension three then, the Sridharan enveloping algebra $\mathcal{U}_f(\mathcal{G})$, for $f \in Z^2(\mathcal{G}, \mathbb{K})$, is isomorphic to one of ten following associative $\mathbb{K}$-algebras, defined by three generator x, y, z and the following commutation relations (see [32], Theorem 1.3):

Type	$[x,y]$	$[y,z]$	$[z,x]$
1	0	0	0
2	0	x	0
3	x	0	0
4	0	αy	$-x$
5	0	y	$-(x+y)$
6	z	$-2y$	$-2x$
7	1	0	0
8	1	x	0
9	x	1	0
10	1	y	x

where $\alpha \in \mathbb{K} - \{0\}$. Therefore the Sridharan enveloping algebra $\mathcal{U}_f(\mathcal{G})$ is a skew *PBW* extension.

Let $\mathcal{G}$ be a finite dimensional Lie algebra. Then for any 2$-$cocycle $f \in Z^2(\mathcal{G}, \mathbb{K})$, the following statements are equivalent (see [22], Theorem 5.3).

(i) The Sridharan enveloping algebra $\mathcal{U}_f(\mathcal{G})$ is Calabi-Yau of dimension d.

(ii) The universal enveloping algebra $\mathcal{U}(\mathcal{G})$ is Calabi-Yau of dimension d.

Let $\mathcal{U}_f(\mathcal{G})$ be a Sridharan enveloping algebra of a finite dimensional Lie algebra $\mathcal{G}$. Then $\mathcal{U}_f(\mathcal{G})$ is Calabi-Yau of dimension 3 if and only if $\mathcal{U}_f(\mathcal{G})$ is isomorphic to $\mathbb{K}\langle x,y,z\rangle/\langle R\rangle$ with the commuting relations R listed in the following table (see [22], Theorem 5.5):

Case	{x,y}	{x,z}	{y,z}
1	z	$-2x$	$2y$
2	y	$-z$	0
3	z	0	0
4	0	0	0
5	y	$-z$	1
6	z	1	0
7	1	0	0

where $\{x,y\} = xy - yx$.

From the above discussion we have the following result.

Proposition 2. *Let $\mathcal{U}_f(\mathcal{G})$ be a Sridharan enveloping algebra of a finite dimensional Lie algebra $\mathcal{G}$. If $\mathcal{U}_f(\mathcal{G})$ is Calabi-Yau of dimension 3 then $\mathcal{U}(\mathcal{G})$ is a skew PBW extension.*

The Sridharan enveloping algebra of an $n-$dimensional abelian Lie algebra is $n-$Calabi-Yau; in particular the Weyl algebra A_n is $2n-$Calabi-Yau (see [9], Theorem 6.5) as well as a skew PBW extension (see [16], Example 5).

3.2. Counterexamples

Next we will show some examples of algebras that are AS-regular, or N-Koszul, or Calabi-Yau, but are not skew PBW extensions.

1. $A = \mathbb{K}\langle x,y,z\rangle/\langle xy-yx-z^2, yz-z-x^2, zx-xz-y^2\rangle$ is AS-regular of global dimension 3 of type $\mathtt{A}$ (see [2], page 173). A is 2-Koszul (see [8], Proposition 5.2) and Calabi-Yau of dimension 3 (see [6], Proposition 5.4).
2. $A = \mathbb{K}\langle x,y\rangle/\langle x^3+xy^2+y^2x+yxy, x^2y+yx^2+xyx+y^3\rangle$ is AS-regular of global dimension 3 of type $\mathtt{A}$ (see [2], Theorem 3.10), A is 3-Koszul (see [8], Proposition 5.2) and Calabi-Yau of dimension 3 (see [6], Proposition 5.4).
3. $A = \mathbb{K}\langle x,y\rangle/\langle yx\rangle$ is not AS-regular algebra. A is the only graded algebra of global dimension 2 and GK-dimension 2 which is not Noetherian (see [2], page 172). A is 2-Koszul (see [15], page 7), A is not 2-Calabi-Yau (see [6], Proposition 4.3)
4. The exterior algebra $A = \mathbb{K}\langle x_1,\cdots,x_n\rangle/\langle x_k^2, x_ix_j + x_jx_i\rangle_{k,i<j}$ in n variables is an 2-Koszul algebra (see [31], Example 1.6).
5. If $A = \mathbb{K}\langle x_1,\cdots,x_n\rangle/I$ is an quadratic algebra and I is principal, then A is 2-Koszul (see [15], page 7). It depends on the ideal I whether A is Calabi-Yau or not.
6. Consider V of dimension 1, $V = \mathbb{K}x$ and $w = x^{N+1}$. Then, $dimR = dimR_{N+1} = 1$, $A(w)$ is N-Koszul (since the global dimension of $A(w)$ is infinite, and $A(w)$ is not 3-Calabi-Yau (see [4], Example 2.12).

4. Some Properties Preserved by Ore Extensions

Suppose $\sigma : A \to A$ is a graded algebra automorphism and $\delta : A(-1) \to A$ is a graded σ-derivation. If $B := A[z;\sigma,\delta]$ is the associated Ore extension, then B is a skew PBW extension. In this case we have $B = A[z,\sigma;\delta] = \sigma(A)\langle x\rangle$ (see [16], Example 5).

Below we list some properties that are preserved by Ore extensions:

1. If A is a connected graded algebra then B is a connected graded algebra.
2. If A is homologically smooth, then so is B (see [30], Proposition 3.1).
3. B is 2-Koszul if and only if A is 2-Koszul (see [33], Corollary 1.3).
4. Let $A = \mathbb{K}\langle x_1,\dots,x_n\rangle/\langle f\rangle$ where $f = (x_1,\dots,x_n)M(x_1,\cdots,x_n)^t$ and M is an $n\times n$ matrix. Then A is Calabi-Yau of dimension 2 if and only if M is invertible and anti-symmetric (see [24], Corollary 1).

   Let δ be a graded derivation of the free algebra $\mathbb{K}\langle x_1,\dots,x_n\rangle$ of degree 1. If $\delta(f) = 0$, then δ

induces a graded derivation $\overline{\delta}$ on A. Let $B = A[z;\overline{\delta}]$ be the Ore extension of A defined by the graded derivation $\overline{\delta}$. Then B is a graded Calabi-Yau algebra of dimension 3 (see [21], Proposition 1.3).

5. If A is $\nu-$skew Calabi-Yau projective $\mathbb{K}$-algebra of dimension d, then B is skew Calabi-Yau of dimension $d+1$ and the Nakayama automorphism ν' of B satisfies that $\nu'_{|A} = \sigma^{-1}\nu$ and $\nu'(z) = uz+b$, with $u, b \in A$ and u invertible (see [30], Theorem 3.3).

6. Let A be a 2-Koszul AS-regular algebra of global dimension d with the Nakayama automorphism ξ. Then $B = A[z,\xi]$ is a Calabi-Yau algebra of dimension $d+1$ (see [25], Theorem 3.3).

7. Let A be a ν-skew Calabi-Yau algebra of dimension d and $\sigma \in Aut(A)$, then $A[x;\sigma]$ and $A[x^{\pm 1};\sigma]$ are Calabi-Yau algebras of dimension $d+1$ (see [18], Theorema 1.1). Furthermore, if $A[x;\sigma]$ is Calabi-Yau, then $A[x^{\pm 1};\sigma]$ is Calabi-Yau.

8. Now we present an example of skew Calabi-Yau algebra that is not Calabi-Yau (see [30]), and then, we consider the corresponding Ore extension. Let $A = \mathbb{K}\langle x,y\rangle/\langle yx - xy - x^2\rangle$ be the Jordan plane, A is AS-regular algebra of dimension 2 and therefore A is 2-Koszul, $A = \mathbb{K}[x][y,\delta_1]$ with $\delta_1(x) = x^2$. It follows that A is skew Calabi-Yau but not Calabi-Yau. A has Nakayama automorphism given by $\nu(x) = x$ and $\nu(y) = 2x+y$, $B = A[z;\nu]$ is an Ore extension of Jordan plane. Then B is skew Calabi-Yau with the Nakayama automorphism ν' such that $\nu'(x) = x$ and $\nu'(y) = y$. $B = \mathbb{K}[x,z][y;\delta]$ where δ is given by $\delta(x) = x^2$ and $\delta(z) = -2xz$. So, $\nu'(z) = z$. It follows that B is Calabi-Yau, which was already proved by Berger and Pichereau in [5].

9. In [44], AS-regular algebras of dimension 5 generated by two generators of degree 1 with three generating relations of degree 4 are classified under some generic condition. There are nine types such AS-regular algebras in this classification list. Among them, the algebras **D** and **G** are given by iterated Ore extensions (see [44], Section 5.2).

   The algebra **D** is skew Calabi-Yau with the Nakayama automorphism ν given by $\nu(x) = p^{-3}q^4x$; $\nu(y) = p^3q^{-4}y$. **D** is Calabi-Yau if and only if that p,q satisfy the system of equations (see [30], Theorem 4.3):

$$\begin{cases} p^3 = q^4, \\ 2p^4 - p^2q + q^2 = 0. \end{cases}$$

   The algebra **G** is skew Calabi-Yau with the Nakayama automorphism ν given by $\nu(x) = gx$; $\nu(y) = g^{-1}y$. **D** is Calabi-Yau if and only if $g = 1$.

   They study and classification of AS-regular algebras of dimension five with two generators under an additional $\mathbb{Z}^2$-grading uses Gröbner basis computations (see [48]).

10. Let $\mathbb{K}$ be a field, let n be an even natural number ≥ 2, and let A be the associative $\mathbb{K}$-algebra defined by generators $x_1, \ldots, x_n$ subject to the single relation

$$\sum_{1\leq i\leq \frac{n}{2}} [x_i, x_i + \frac{n}{2}] = \nu + \lambda,$$

   where the bracket stands for the commutator, ν is a linear combination of the x_i's, and $\lambda \in \mathbb{K}$. Then the filtered algebra A is 2-Koszul. Furthermore A is 2-Calabi-Yau if and only if $\nu = 0$ (see [9], Theorem 6.4). So, if $\sigma_2 = i_{\mathbb{K}[x_1]}$ and $\delta_2(\mathbb{K}[x_1]) \subseteq \mathbb{K}$, then the skew PBW extension $\sigma(\mathbb{K})\langle x_1, x_2\rangle \cong \mathbb{K}[x_1][x_2;\sigma_2,\delta_2]$ is 2-Calabi-Yau.

References

[1] J. P. Acosta, C. Chaparro, O. Lezama, I. Ojeda and C. Venegas, “Ore and Goldie theorems for skew PBW extensions”, *Asian-European J. Math.*, vol. 6 (4), pp. 1350061-1 - 1350061-20, 2013.

[2] M. Artin and W. F. Schelter, “Graded algebras of global dimension 3”, *Adv. Math.*, vol. 66, pp. 171-216, 1987.

[3] R. Berger, “Koszulity for nonquadratic algebras”, *J. Algebra*, vol. 239, pp. 705-734, 2001.

[4] R. Berger and A. Solotar, “A criterion for homogeneous potencials to be 3-Calabi-Yau”, *arXiv:1203.3029*, 2013.

[5] R. Berger and A. Pichereau, "Calabi-Yau algebras viewed as deformations of Poisson algebras", *Algebr. Represent. Theory*, vol. 17 (3), pp. 735-773, 2014.

[6] R. Berger and R. Taillefer, "Poincare-Birkhoff-Witt deformations of Calabi-Yau algebras", *J. Noncommut. Geom.*, vol. 1, pp. 241-270, 2007.

[7] R. Berger and V. Ginzburg, "Higher Symplectic Reflection Algebras and Non-homogeneous N-Koszul Property", *J. Algebra*, vol 304, pp. 577-601, 2006.

[8] R. Berger and N. Marconnet, "Koszul and Gorenstein properties for homogeneous algebras", *Algebr. Represent. Theory*, vol. 1, pp. 67-97, 2006.

[9] R. Berger, "Gerasimov's theorem and N-Koszul algebras", *J. London Math. Soc.*, vol. 79, pp. 631-648, 2009.

[10] R. Bocklandt, "Graded Calabi Yau algebras of dimension 3 (with an appendix by M. Van den Bergh)", *J. Pure Appl. Algebra*, vol. 212, pp. 14-32, 2008.

[11] R. Bocklandt, "Calabi Yau algebras and weighted quiver polyhedra", *Math. Z.*, vol. 273, pp. 311-329, 2013.

[12] K. A. Brown, I. G. Gordon and C. H. Stroppel, "Cherednik, Hecke and quantum algebras as free Frobenius and Calabi-Yau extensions", *J. Algebra*, vol. 319 (3), pp. 1007-1034, 2008.

[13] T. Cassidy and B. Shelton, "Generalizing the notion of Koszul algebra", *Math. Z.*, vol. 260, pp. 93-114, 2008.

[14] G. Floystad and J. E. Vatne, "PBW-Deformations of N-Koszul Algebras", *J. Algebra*, vol. 302, pp. 116-155, 2006.

[15] R. Fröberg, "Koszul Algebras", *Matematiska institutionen Stockholms Universitet 10691 Stockholm Sweden*, 1998.

[16] C. Gallego and O. Lezama, "Gröbner bases for ideals of $\sigma - PBW$ extensions", *Comm. Algebra*, vol. 39, pp. 1-26, 2011.

[17] V. Ginzburg, "Calabi-Yau algebras", *arXiv:math.AG/0612139v3*, 2006.

[18] J. Goodman, and U. Krähmer, "Untwisting a twisted Calabi-Yau Algebra", *J. Algebra*, vol. 406, pp. 271-289, 2014.

[19] E. L. Green and E. Marcos, "d-Koszul algebras, 2-d determined algebras and 2-d-Koszul algebras", *J. Pure Appl. Algebra*, vol. 215 (4), pp. 439-449, 2011.

[20] E. L. Green, E. Marcos, R. Martínez-Villa and P. Zhang, "D-Koszul algebras", *J. Pure Appl. Algebra*, vol. 193 (1-3), pp. 141-162, 2004.

[21] J. W. He, F. Van Oystaeyen and Y. Zhang, "Graded 3-Calabi-Yau Algebras as Ore Extensions of 2-Calabi-Yau Algebras", *arXiv:1303.5293v1*, 2013.

[22] J. W. He, F. Van Oystaeyen and Y. Zhang, "Cocommutative Calabi-Yau Hopf Algebras and Deformations", *J. Algebra*, vol. 324, pp. 1921-1939, 2010.

[23] J. W. He, F. Van Oystaeyen and Y. Zhang, "Hopf algebra actions on differential graded algebras and applications", *Bull. Belg. Math. Soc. Simon Stevin*, vol. 18, pp. 99-111, 2011.

[24] J. W. He, F. Van Oystaeyen and Y. Zhang, "Calabi-Yau algebras and their deformations", *Bull. Math. Soc. Sci. Math. Roumanie*, Tome 56 (104), no. 3, pp. 335-347, 2013.

[25] J. W. He, F. Van Oystaeyen and Y. Zhang, "Skew polynomial algebras with coeficients in Koszul Artin-Schelter regular algebras", *J. Algebra*, vol. 390, pp. 231-249, 2013.

[26] J. W. He, F. Van Oystaeyen and Y. Zhang, "Deformations of Koszul Artin-Schelter Gorenstein algebras", *Manuscripta Math., Doi: 10.1007/s00229-012-0580-z*, 2012.

[27] O. Iyama, I. Reiten, "Fomin-Zelevinsky mutation and tilting modules over Calabi-Yau algebras", *Am. J. Math.*, vol. 130, pp. 1087-1149, 2008.

[28] P. Le Meur, "Crossed productd of Calabi-Yau algebras by finite groups", *arXiv:1006.1082*, 2010.

[29] O. Lezama and A. Reyes, ´´Some Homological Properties of Skew *PBW* Extensions", *Comm. Algebra*, vol. 42, pp. 1200-1230, 2014.

[30] L. -Y. Liu, S. Wan and Q. -S Wu, "Twisted Calabi-Yau property of Ore extensions", *J. Noncommut. Geom.*, vol. 8 (2), pp. 587-609, 2014.

[31] R. Martínez-Villa, "Introduction to Koszul Algebras", *Rev. Un. Mat. Argentina*, vol. 48 (2), pp. 67-95, 2007.

[32] P. Nuss, "L'homologie cyclique des algèbres enveloppantes des algèbres de Lie de dimension trois", *J. Pure Appl. Algebra*, vol. 73, pp. 39-71, 1991.

[33] C. Phan, "The Yoneda algebra of a graded Ore extension", *Comm. Algebra*, vol. 40, pp. 834-844, 2012.

[34] S. Priddy, "Koszul Resolutions", *Trans. Amer. Math. Soc.*, vol. 152, pp. 39-60, 1970.

[35] A. Reyes, "Uniform Dimension over Skew PBW Extensions", *Rev. Colombiana Mat.*, vol. 48, pp. 79-96, 2014.

[36] A. Reyes, "Gelfand-Kirillov Dimension of Skew PBW Extensions", *Rev. Colombiana Mat.*, vol. 47 (1), pp. 95-111, 2013.

[37] M. Reyes, D. Rogalski and J. Zang, "Skew Calabi-Yau Algebras and Homological Identities", *arXiv:1302.0437v2 [math.RA]*, 2013.

[38] B. Shelton and C. Tingey, "On Koszul algebras and a new construction of Artin-Schelter regular algebras", *J. Algebra*, vol. 241 (2), pp. 789-798, 2001.

[39] S. P. Smith, "A 3-Calabi-Yau algebra with G_2 symetry constructed from the octonions", *arXiv:1104.3824v1*, 2011.

[40] D. R. Stephenson and J.J. Zhang, "Growth of graded Noetherian rings", *Proc. Amer. Math. Soc.*, vol. 125, pp. 1593-1605, 1997.

[41] M. Suárez-Alvarez, "Untwisting algebras with van den Bergh duality into Calabi-Yau algebras", *arXiv:1311.3339v1*, 2013.

[42] M. Van den Bergh, "Introduction to Super Potentials (informal evening talk)", *Oberwolfach Report 6*, 2005.

[43] M. Van den Bergh, "Calabi-Yau algebras and superpotentials", *Sel.Math.New Ser DOI 10.1007/s00029-014-0166-6*, 2014.

[44] S. -Q Wang and Q. -S Wu, "A class of AS-regular algebras of dimension five", *J. Algebra*, vol. 362, pp. 117-144, 2012.

[45] Q. -S. Wu and C. Zhu, "Skew group algebras of Calabi-Yau algebras", *J. Algebra*, vol. 340, pp. 53-76, 2011.

[46] Q. -S. Wu and C. Zhu, "Poincaré-Birkhoff-Witt Deformation of Koszul Calabi-Yau Algebras", *Algebr. Represent. Theory*, vol. 16, pp. 405-420, 2013.

[47] X. Yu and Y. Zhang, "Calabi-Yau pointed Hopf algebras of finite Cartan type", *J. Noncommut. Geom.*, vol. 7 (4), pp. 1105-1144, 2013.

[48] G. -S. Zhou and D. -M Lu, "Artin-Schelter regular algebras of dimension five with two generators", *J. Pure Appl. Algebra*, vol. 218, pp. 937-961, 2014.

[49] C. Zhu, F. Van Oystaeyen and Y. Zhang, "Calabi-Yau extensions and localization of Koszul regular algebras", *arXiv:1401.0330v1*, 2014.