

Dynamics of a System Associated with a Piecewise Quadratic Family

Dinámica de un sistema asociado a una familia cuadrática a tramos

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Abstract

This paper presents a study, both in analytical and numerical form, of a discrete dynamical system associated with a piecewise quadratic family. The orbits of periods one and two were characterized, and their stability was established. The nonsmooth phenomenon known as border collision is present when there is a period doubling. Lyapunov exponents are calculated numerically to determine the presence of chaos in the system.

Key words: Bifurcation, Chaos, Stability, Periodic orbit, Fixed point, Dynamical system.

Resumen

Presenta un estudio analítico y numérico de la dinámica de un sistema discreto asociado a una familia cuadrática a tramos; se caracterizan las órbitas de período uno y dos, así como su estabilidad; se muestra la presencia del fenómeno no suave, conocido como bifurcación por colisión de borde cuando ocurre un doblamiento de período. Se hallaron numéricamente los exponentes de Lyapunov para detectar la presencia de caos en el sistema.

Palabras clave: Bifurcaciones, Caos, estabilidad, Órbita periódica, Punto fijo, Sistema dinámico.

1. Introduction

Physical phenomena such as impacting oscillators and switched electronic circuits, among others, have stimulated the study of piecewise-smooth dynamical systems. These systems are characterized by the partition of the state space into regions deter-

mined by smooth vector spaces, but display discontinuities on the borders between those regions, which turn the state space into a non-smooth vector space, as has been studied by several authors [1]-[4]. This non-smoothness accounts for the induction of phenomena such as border-collision bifurcations, which

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appear in certain discrete dynamical systems [5]-[7] and continuous systems [8]-[9], and more complex systems such as Boost DC-DC converters, where a turn on-turn off switching action partitions the state space into two regions. Border collision bifurcations appear in some systems where bifurcation due to period doubling is also present [10]-[13], which is the situation studied in this paper.

The paper is organized as follows: in Section 2, the fixed points, the 2-cycles and their respective attraction bases, are found analytically. In Section 3, the bifurcation diagram is found, and the border collision bifurcation present in the system whenever a period doubling occurs is analyzed. A comparative analysis is also performed for the bifurcations in the system that are attributable to the usual period doubling. In Section 4, the Lyapunov exponents are found to determine the presence of chaos in the system.

2. Dynamic Model and Fixed Points

This paper consider the dynamical system $X_{n+1} = f(X_n)$ associated with the function:

$$f(x) = \begin{cases} \mu(1-x^2) & \text{if } x \leq 0 \\ \mu(x^2-1) & \text{if } x > 0 \end{cases}$$

and we take $\mu > 0$ to be the bifurcation parameter.

We start by recalling that x is a virtual fixed point of f if $f(x) = x$ and x does not belong to the domain of f , as discussed by O. Eriksson *et al.* [14]. If a point x exists such that $f(x) = x$, and x belongs to the domain of f then x is said to be a fixed point, and its orbit consists of a single element. On the other hand, if $n \in \mathbb{N}$ exists such that $f^n(x) = x$ and $f^k(x) \neq x$, for each $0 < k < n$, then x is a periodic point of f with period n . If x is a periodic point of period $n \in \mathbb{N}$ then the orbit of x consists of exactly the n elements $\{x, f(x), f^2(x), \dots, f^{n-1}(x)\}$.

In the case where the positive orbit of a point y is infinite, the question arises whether the sequence of iterations of y converges to a periodical point of the function, in addition to the influence of the parameters of the function on such a convergence. As usual, the set of points of the domain of f whose orbit converges to a fixed point x is called the basin of attraction of x and is denoted by B_x . If that set

contains only the point x it is said that x is a repeller or unstable fixed point. Otherwise, x is called an attractor or stable fixed point.

2.1. Fixed Points

For $x > 0$, f has a fixed point x_1 and a virtual fixed point x_2 :

$$x_2 = \frac{1 - \sqrt{1 + 4\mu^2}}{2\mu} \quad x_1 = \frac{1 + \sqrt{1 + 4\mu^2}}{2\mu}$$

with $-1 < x_2 < 0$ and $1 < x_1$.

For $x \leq 0$, f has a fixed point z_1 and a virtual fixed point z_2 , where $z_1 = -x_1$ and $z_2 = -x_2$, such that:

$$z_1 < -1 \quad \text{and} \quad 0 < z_2 < 1$$

By the derivative criterion x_1 and z_1 are unstable.

In Figure 1 the fixed points for $f(x)$ may be observed.

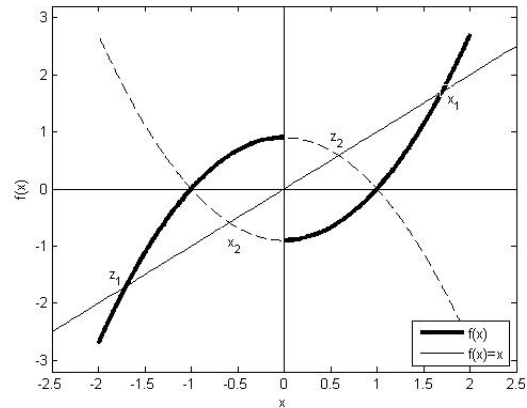


Figure 1. $f(x)$ with $\mu = 0.9$.

3. Results and Discussion

3.1. Orbits of Period Two

To find the orbits of period two, or 2-cycles, we set $f^2(x) = x$. From the definition of f , it can be seen that

$$f^2(x) = \begin{cases} g(x) = \mu - \mu^3(1-x^2)^2 & \text{if } x \in (-\infty, -1] \cup (0, 1] \\ h(x) = \mu^3(1-x^2)^2 - \mu & \text{if } x \in (-1, 0] \cup (1, +\infty) \end{cases}$$

The virtual fixed points of $f(x)$, $\Lambda_1 = \{x_2, z_2\}$, form a 2-cycle as shown in Figure 2. Note that $f^2(x_2) = h(x_2) = x_2$ and $f^2(z_2) = g(z_2) = z_2$

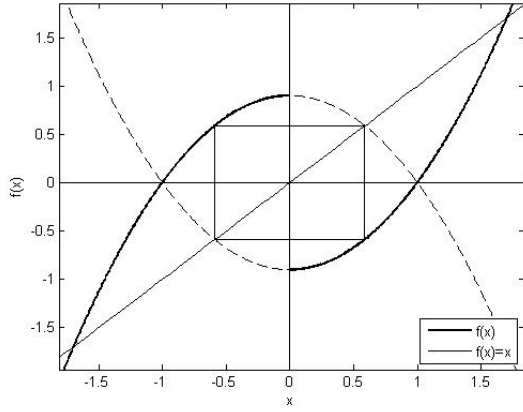


Figure 2. Λ_1 for $\mu = 0.9$.

By the derivative criterion we have that if $\frac{\sqrt{3}}{2} < \mu$, the 2-cycle Λ_1 is unstable, and if $0 < \mu \leq \frac{\sqrt{3}}{2}$, the 2-cycle is stable.

To find the basin of attraction of Λ_1 we require the fact that f^2 is increasing, as well as the following lemmas.

Lemma 1. *If $x \in (0, z_2)$ then $g^n(x) \in (0, z_2)$ for all $n \in \mathbb{N}$ and the sequence $(g^n(x))_n$ is increasing in this interval.*

As $0 < x < z_2$, then $0 < g(0) = \mu - \mu^3 < g(x) < g(z_2) = z_2$ and by induction, it follows that $0 < g^n(x) < z_2$. Now, because

$$\begin{aligned} g(x) - x &= -\mu^3 x^4 - 2\mu^3 x^2 - x + \mu - \mu^3 \\ &= (x - z_1)(x - z_2)(-\mu^3 x^2 + \mu^2 x - \mu + \mu^3) \end{aligned}$$

then $z_1 < 0 < x < z_2$ and $(\forall x \in \mathbb{R})(-\mu^3 x^2 + \mu^2 x - \mu + \mu^3 < 0)$ we have that $g(x) - x > 0$. Hence, because f^2 is increasing, g is also increasing; consequently $0 < g^n(x) < g^{n+1}(x) < z_2$

Lemma 2. *If $x \in (0, z_2)$ then $g^n(x) \rightarrow z_2$.*

By Lemma 1, there is an $L > 0$ such that $\lim_{n \rightarrow \infty} g^n(x) = L$. Suppose that $L < z_2$. Because $L < g(L)$ and $g(L) < \mu$ then $L < \mu$. Let

$$q = \sqrt{1 - \sqrt{\frac{\mu - L}{\mu^3}}}$$

Then $q < g(q) = L$. For $\varepsilon = L - q$ there exists $N \in \mathbb{N}$ such that $\forall n > N \quad |g^n(x) - L| < \varepsilon$. Thus, $L - g^n(x) = |g^n(x) - L| < \varepsilon$, $0 < q < g^n(x) < z_2$ and because g is increasing, then $L < g^{n+1}(x)$ for $x \in (0, z_2)$, which is a contradiction. Hence, $\lim_{n \rightarrow \infty} g^n(x) = z_2$.

Proposition 1. $(0, 1] \subseteq B_{z_2}$

By Lemma 2 we have that $(0, z_2] \subseteq B_{z_2}$. In an analogous way, it is shown that, if $x \in (z_2, 1]$ then $g^n(x) \rightarrow z_2$.

To find the remaining intervals of the basin of attraction of z_2 , we define the function l with domain $\mathbb{R}^+ \cup \{0\}$ as $l(z) = \sqrt{1 + \sqrt{\frac{\mu+z}{\mu^3}}}$. We have that $l(z) = h^{-1}(z)$ for $z > 1$ and $l(z)$ is increasing in $z \in (1, x_1)$. So, it can be seen that the function l satisfies the relation $1 < l(0) < l(1)$. From this inequality, and using the fact that l is increasing, it follows that:

$$\begin{aligned} 1 < l(0) < l(1) < l^2(0) < l^2(1) < \dots \\ \dots < l^n(0) < l^n(1) < \dots \end{aligned} \quad (1)$$

Now, we will prove that $l^n(1) \rightarrow x_1$ whenever $n \rightarrow +\infty$. Because $l^n(1)$ is increasing and bounded, there exists an L such that $\lim_{n \rightarrow \infty} l^n(1) = L$. Assume that $L < x_1$ and let $\varepsilon = L - h(L) > 0$. There exists an $N \in \mathbb{N}$ such that $\forall n > N, |l^n(1) - L| < \varepsilon$. Hence $L - l^n(1) < L - h(L)$ and applying h we have that $0 < h(L) < l^n(1) < x_1$. So $0 < L < l^{n+1}(1)$ which is a contradiction.

In an analogous way, it may be seen that $l^n(0) \rightarrow x_1$.

From the last two observations, it follows that the length of the intervals $[l^n(0), l^n(1)]$ tends to zero when $n \rightarrow +\infty$.

Proposition 2. $\bigcup_{n \in \mathbb{N}} [l^n(0), l^n(1)] \subseteq B_{z_2}$.

Let $x \in \bigcup_{n \in \mathbb{N}} [l^n(0), l^n(1)]$ and $x \in [l^k(0), l^k(1)]$ for some $k \in \mathbb{N}$. Then, $l^{k-1}(0) < h(x) \leq l^{k-1}(1)$ and because h is increasing, applying h in the last inequality $k - 1$ times results in $0 < h^k(x) \leq 1$. By Proposition 1, it follows that $h^k(x) \in B_{z_2}$.

A similar line of reasoning leads to the following result:

Proposition 3. $\bigcup_{n \in \mathbb{N}} (-l^{n+1}(0), -l^n(1)] \subseteq B_{z_2}$.

From propositions 1, 2 and 3 we have that

$$C = \left(\bigcup_{n \in \mathbb{N}} (-l^{n+1}(0), -l^n(1)] \right) \cup (0, 1] \cup \left(\bigcup_{n \in \mathbb{N}} [l^n(0), l^n(1)] \right) \subseteq B_{z_2}$$

In fact $C = B_{z_2}$, as we shall see further on.

Proposition 4. $D = (\bigcup_{n \in \mathbb{N}} (-l^n(1), -l^n(0))) \cup (-1, 0] \cup (\bigcup_{n \in \mathbb{N}} (l^n(1), l^{n+1}(0))) \subseteq B_{x_2}$.

Using mathematical induction, we see that $-f^n(x) = f^n(-x)$. Now, let $x \in B_{z_2}$ and we shall show that $-x \in B_{x_2}$ for all $n \in \mathbb{N}$ with $x \neq l^n(1)$ and $x \neq l^n(-1)$. If $x \in B_{z_2}$ then $f^n(x) \rightarrow z_2$, $-f^n(x) \rightarrow -z_2$ and therefore $f^n(-x) \rightarrow x_2$. If $x = 0$ then $f(0) = \mu^3 - \mu$ and because $-1 < \mu^3 - \mu < 0$, we have that $0 \in B_{x_2}$.

Finally, if $x \in D$ then $-x \in C$ so $-x \in B_{z_2}$, leaving us with $x \in B_{x_2}$.

We shall now look at the basin of attraction of the 2-cycle Λ_1 .

Proposition 5. $B_{\Lambda_1} = B_{x_2} \cup B_{z_2} = (z_1, x_1)$ with $B_{x_2} \cap B_{z_2} = \emptyset$.

Because $B_{x_2} \subseteq (z_1, x_1)$ and $B_{z_2} \subseteq (z_1, x_1)$, it follows that $B_{x_2} \cup B_{z_2} \subseteq (z_1, x_1)$. Now, let $x \in (z_1, x_1)$ and consider the case where $x \in (0, x_1)$. For $\varepsilon = x_1 - x$ there exists $n_1 \in \mathbb{N}$ such that, for all $n > n_1$, $|l^n(1) - x_1| < \varepsilon$. Hence $x < l^n(1)$ and consequently, $x \in B_{x_2} \cup B_{z_2}$.

In a similar way, the inclusion for the case $x \in (z_1, 0]$ may be shown.

Figure 3 shows the basin of attraction of Λ_1 obtained numerically.

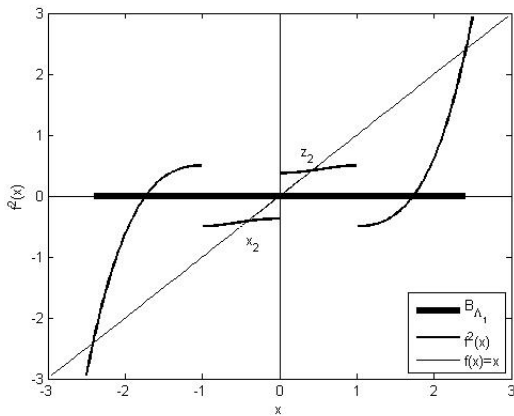


Figure 3. B_{Λ_1} for $\mu = 0.5$.

Next, we characterize the remaining 2-cycles of the system. For this purpose, let $h(x) = x$, initially, that is:

$$\mu^3(x - x_1)(x - x_2)(x - x_3)(x - x_4) = 0$$

where $x_3 = \frac{-1 - \sqrt{4\mu^2 - 3}}{2\mu}$ and $x_4 = \frac{-1 + \sqrt{4\mu^2 - 3}}{2\mu}$. For

$g(x)$ the fixed points are $z_1, z_2, z_3 = -x_3$ and $z_4 = -x_4$.

If $0 < \mu < \frac{\sqrt{3}}{2}$, then x_3, x_4, z_3 and z_4 are not real. Hence, there are no more 2-cycles.

If $\mu = \frac{\sqrt{3}}{2}$, then $x_3 = x_4 = x_2 = \frac{-1}{2\mu}$ and $z_3 = z_4 = z_2 = \frac{1}{2\mu}$, and we again obtain the 2-cycle Λ_1 .

If $\frac{\sqrt{3}}{2} < \mu < 1$, we have $-1 < x_3 < x_4 < 0$ and $0 < z_4 < z_3 < 1$. So x_3 and x_4 belong to the domain of $h(x)$ and z_3, z_4 belong to the domain of $g(x)$.

Consequently, we have the following additional 2-cycles:

$$\Lambda_2 = \{x_3, z_4\} \quad \text{y} \quad \Lambda_3 = \{x_4, z_3\}$$

By the derivative criterion, both 2-cycles are stable. The 2-cycles Λ_2 and Λ_3 are shown in Figure 4.

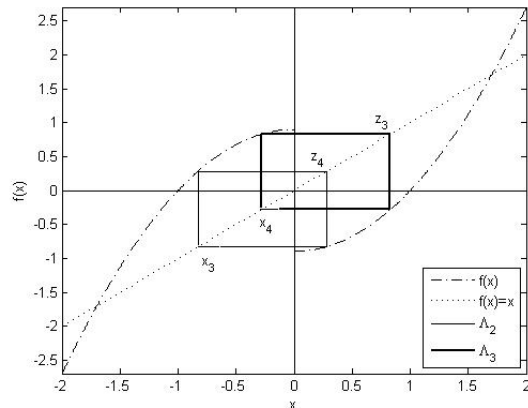


Figure 4. Λ_2 and Λ_3 for $\mu = 0.9$.

A similar construction for the basin of Λ_1 leads to:

$$B_{x_3} = \left(\bigcup (-l^n(1), -l^n(-x_2)) \right) \cup (-1, x_2) \cup \left(\bigcup (l^n(1), l^{n+1}(x_2)) \right)$$

$$B_{z_4} = \left(\bigcup (-l^n(0), -l^n(x_2)) \right) \cup (0, z_2) \cup \left(\bigcup (l^n(0), l^n(z_2)) \right)$$

$$B_{x_4} = \left(\bigcup (-l^n(-x_2), -l^n(0)) \right) \cup (x_2, 0) \cup \left(\bigcup (l^n(x_2), l^n(0)) \right)$$

$$B_{z_3} = \left(\bigcup (-l^{n+1}(x_2), -l^n(1)) \right) \cup (z_2, 1) \cup \left(\bigcup (l^n(z_2), l^n(1)) \right)$$

So, $B_{\Lambda_2} = B_{x_3} \cup B_{z_4}$ and $B_{\Lambda_3} = B_{x_4} \cup B_{z_3}$, as illustrated in Figures 5 and 6.

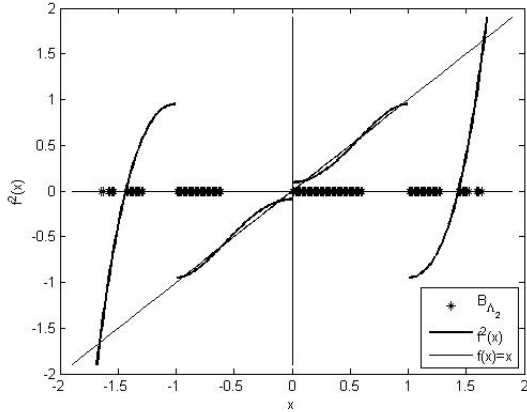


Figure 5. B_{Λ_2} for $\mu = 0.95$.

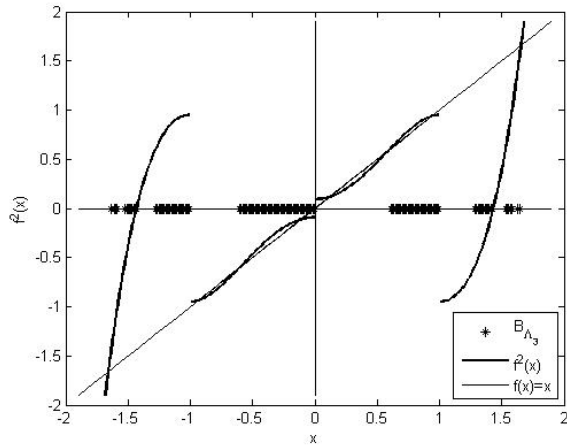


Figure 6. B_{Λ_3} for $\mu = 0.95$.

If $\mu = 1$, then $x_3 = -1$, $x_4 = 0 = z_4$ and $z_3 = 1$, in which case the fixed point x_3 does not belong to the domain of $h(x)$; nevertheless, we have found the stable 2-cycle $\Lambda_4 = \{0, 1\}$ whose basin of attraction is $B_0 \cup B_1$, where

$$B_0 = \left(\bigcup (-l^n(-x_2), -l^n(x_2)) \right) \cup (x_2, z_2) \cup \left(\bigcup (l^n(x_2), l^n(z_2)) \right)$$

$$B_1 = \left(\bigcup (-l^{n+1}(x_2), -l^n(1)) \right) \cup (z_2, 1] \cup \left(\bigcup (l^n(z_2), l^n(1)) \right)$$

If $1 < \mu$, then $x_3 < -1$, $0 < x_4 < 1$, $1 < z_3$ and $-1 < z_4 < 0$. Consequently, these points do not belong to the domains of $h(x)$ and $g(x)$, respectively, so these cases are not considered.

Finally, if $x > x_1$ it may be seen that $f^n(x) \rightarrow +\infty$; additionally, if $x < z_1$ then $f^n(x) \rightarrow -\infty$. Hence,

if $x \in (-\infty, z_1) \cup (x_1, +\infty)$ it follows that x does not belong to the basin of attraction of any 2-cycle.

3.2. Numerical Calculations

To obtain the basins of attraction of Figures 3, 5 and 6 numerically, the following procedure was used: a point x_0 in the domain of f^2 was taken, and its positive orbit $\{x_0, f(x_0), \dots, f^n(x_0)\}$ was found, with $n = 1000$. If the difference between $f^n(x_0)$ and a fixed point x of f^2 was less than $\epsilon = 10^{-3}$ then x_0 was taken as a point in the basin of attraction of x .

3.3. Bifurcation Diagram

It is well-known that the dynamics of a system may change drastically when one or more of its parameters vary. This qualitative or structural change is known as a *bifurcation*. In general, bifurcation theory studies the structural changes undergone by dynamical systems when their parameters change, as discussed elsewhere [15]-[16]. These types of changes may be appreciated in their respective bifurcation diagrams, as the bifurcation diagram determines whether the system converges to an n -cycle or exhibits random behavior. In Figures 7 and 8, a bifurcation diagram is shown for the system under study, where the horizontal axis represents the values of the parameter μ and the vertical axis represents the iterations of the variable x . An initial condition x_0 is chosen, its positive orbit $\{x_0, f(x_0), \dots, f^n(x_0)\}$ is found with $n = 2000$ for each value of μ , and the last 100 iterations are plotted, according [17].

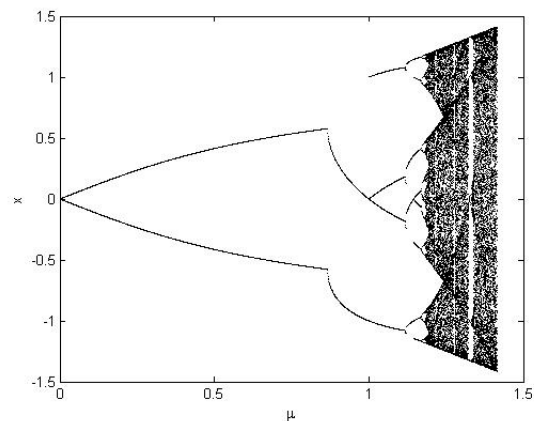


Figure 7. Bifurcation Diagram with $x_0 = 0.5$.

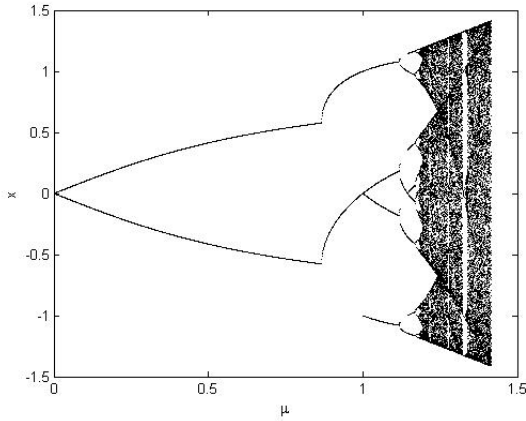


Figure 8. Bifurcation Diagram with $x_0 = -0.5$.

From the diagram, it can be seen that for certain values of the bifurcation parameter, two coexisting stable 2-cycles are present. Then, depending on what basin of attraction the initial condition belongs to, the corresponding 2-cycle will appear in the bifurcation diagram. These diagrams show that, for $0 < \mu < \frac{\sqrt{3}}{2} \approx 0.866$ there exists a stable 2-cycle $\Lambda_1 = \{x_2, z_2\}$ formed by the virtual fixed points of $f(x)$. When $\mu = \frac{\sqrt{3}}{2}$, the first bifurcation occurs, and the former 2-cycle becomes unstable and gives rise to two 2-cycles, Λ_2 and Λ_3 , symmetrical with respect to the point $x = 0$ (as may be seen in Figure 4).

At $\mu = 1$ the second bifurcation appears, called a border-collision bifurcation because when $\mu \rightarrow 1$, then $x_4 \rightarrow 0$ and $z_4 \rightarrow 0$; additionally, at $\mu = 1$, x_4 and z_4 collide at $x = 0$ and disappear to give rise to a 4-cycle. This process is repeated for the bifurcation diagram windows where $f(x)$ does not present chaotic behavior. In Figure 9, an enlargement of the diagram is shown where these two bifurcations occur.

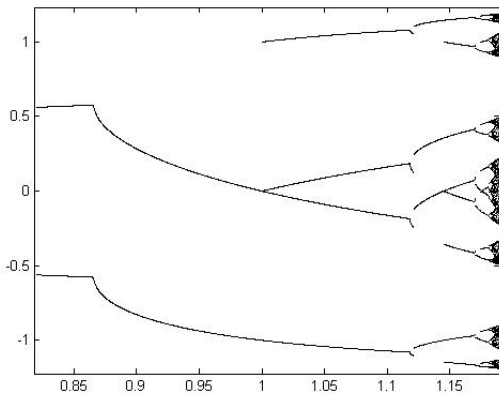


Figure 9. Enlargement of Figure 7.

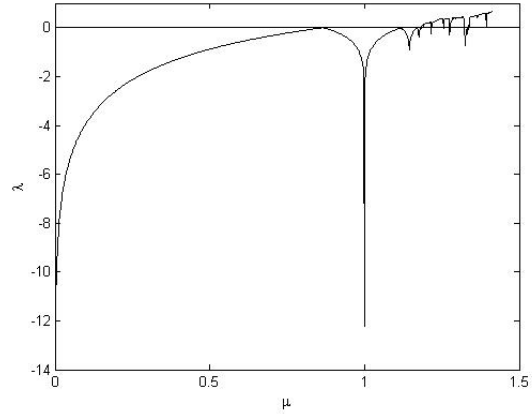


Figure 10. Lyapunov exponents for $f(x)$.

In Figure 11 a comparative analysis is performed between a bifurcation due to period doubling and a bifurcation due to border collision in a scenario with period doubling bifurcation, based on [10].

3.4. Lyapunov Exponents

Lyapunov exponents may be used to measure the future separation of two orbits which were initially very close to each other. In general, the analytic calculation of these exponents is extremely involved, requiring the use of numerical calculation. If the orbits were initially very close to each other and remain so in the future, then the associated Lyapunov exponents will all be negative. However, if the paths diverge, there will be at least one positive Lyapunov exponent. In this manner, Lyapunov exponents are related to the system’s sensitivity to initial conditions.

Lyapunov exponents can be calculated numerically as in [18], with the following method:

$$\lambda(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(x_k)|$$

where $x_k = f^k(x)$.

The Lyapunov exponents for $f(x)$ with the initial condition $x_0 = 0.5$ are shown in Figure 10. It can be seen that, for certain values of the parameter μ , there exist positive Lyapunov exponents. Because the system evolves in a bounded fashion in the state space, it can be concluded that the system exhibits chaotic behavior [17, 19].

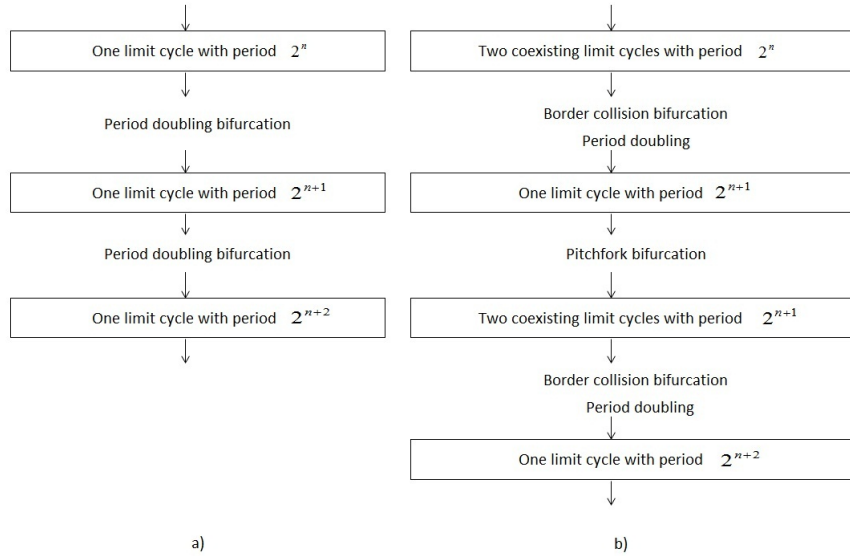


Figure 11. a) Classical period-doubling bifurcation scene. b) Border collision bifurcation in the presence of period doubling.

4. Conclusions

Piecewise smooth one-dimensional applications may generate complex dynamics, such as the coexistence of attractors and border-collision bifurcations, as discussed elsewhere [5, 7].

For the system considered herein, the function $f(x)$ has two unstable fixed points and two virtual fixed points, if $0 < \mu \leq \frac{\sqrt{3}}{2}$, there is only one stable 2-cycle formed by the virtual fixed points; if $\frac{\sqrt{3}}{2} < \mu < 1$, then the existing stable periodic orbit becomes unstable, giving rise to two stable 2-cycles; if $\mu = 1$, the two 2-cycles Λ_2 and Λ_3 collide and form an orbit with period 4. Finally, if $1 < \mu$, the 2-cycle vanishes.

The stability analysis of orbits with periods one and two was performed using the derivative criterion, together with the respective basins of attraction of the orbits, showing that the orbits are formed by a countable collection of intervals whose lengths tend to zero.

Simulations were carried out using Matlab to numerically study the dynamics of the system, and the agreement of simulations with analytic results was confirmed.

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