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Global weak solutions for a 2 × 2 balance non-symmetric system of Keyfitz-Kranzer type

Soluciones débiles globales para un sistema 2 × 2 balanceado no simétrico de tipo Keyfitz-Kranzer

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Abstract

In this paper we consider the Cauchy problem for a particular non-symmetric Keyfitz-Kranzer type system, by using the vanishing viscosity method coupled with the compensated compactness method we get global bounded entropy weak solutions. The main difficulty is to get uniformly bounded estimates on the viscosity method and in this paper is studied.

Keywords**:** Hyperbolic system, global weak solutions, non-symmetric system, Keyfitz-Kranzer type.

Resumen

En este artículo se considera el problema de Cauchy para un sistema no simétrico de tipo Keyfitz-Kranzer y utilizando argumentos de viscosidad nula junto con el método de compacidad compensada se obtiene soluciones débiles entrópicas globales. La principal dificultad es obtener estimaciones uniformemente acotadas en el método de viscosidad y en este trabajo se estudia.

Palabras clave**:** Sistema hiperbólico, soluciones débiles globales, sistema no simétrico , tipo Keyfitz-Kranzer.

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1 Introduction

Many of the natural phenomena are modeled by partial differential equations. Some of the most common partial differential equations are the wave equation, the heat equation, the Schröndinger equation and KdV equation. In this type of problems it is interesting to study the existence and uniqueness of solutions, which in many cases is not an easy task. For example, in the review article [1] the author studies the existence and uniqueness of solutions for non-linear Schröndinger equation. However, in the hyperbolic equations the existence and uniqueness of solutions are a open problems. The goal this paper show the existence of solutions for a hyperbolic system of conservation laws. We consider the balanced non-symmetric system

$$\begin{cases} \rho_t + (\rho\phi(\rho,w)) = f(\rho,w), \\ (\rho w)_t + (\rho w\phi(\rho,w))_x = g(\rho,w) \end{cases} \tag{1}$$

where $\phi(\rho,w) = \Phi(w) - P(\rho)$, Φ a convex function. This system was considered in [2] where the author showed the existence of global weak solution for the homogeneous system (1). Another system of the type (1) was considered in [3] as a generalization to the scalar Buckley-Leverett equations describing two phase flow in porous media. The system (1), recently, has been object of constant studies, in [4] the author considered the particular case in which $\Phi(w) = w$, $P(\rho) = \frac{1}{\rho}$, in this case the two characteristics of the system (1) are linear degenerate, solving the Riemann problem the existence and uniqueness of delta shock solution were established. In this line in [5] the authors considered the case $\Phi(w) = w$ and $P(\rho) = \frac{B}{\rho^\alpha}$ with $\alpha \in (0,1)$, the existence and uniqueness of solutions to the the Riemann problem was got by solving the Generalize Rankine-Hugoniot condition. In both cases, when $\Phi(s) = s$ the system (1) models vehicular traffic flow in a highway without entry neither exit of cars, in this case the source term represents the entry or exit of cars see [6],[7], [8] and reference therein for more detailed description of source term.

Notice that when w is constant, the system (1) reduces to the scalar balance laws

$$\rho_t + (\rho\Phi(w) - \rho P(\rho))_x = f(\rho,w),$$

and from the second equation in (1), g should be of the form

$$g = wf.$$

Moreover, if we make $h(\rho) = \rho(\Phi(w) - P(\rho))$ the global weak solution of the Cauchy problem

$$\begin{cases} \rho_t + h(\rho)_x = f(\rho), \\ \rho(x,0) = \rho_0(x), \end{cases}$$

there exists if $h(\rho)$ is a convex function and the source term is dissipative i.e.,

$$h''(\rho) = -(2P'(\rho) + \rho P''(\rho)) > 0$$
$$sf(s) \le 0.$$

For details on the above result see [9], [10] and references therein. In this work we assume the following conditions,

$\mathscr{C}_1$) f, g are Lipschitz functions such that

$$wf(\rho,w) = g(\rho,w), \quad f(0,0) = 0.$$

$\mathscr{C}_2$) There exist a constant $M > 0$ such that

$$sf(s) \le 0,$$

for $|s| > M$.

$\mathscr{C}_3$) The function $P(\rho)$ satisfies

$$P(0) = 0,\ \lim_{\rho\to 0} \rho P'(\rho) = 0,\ \lim_{\rho\to\infty} P(\rho) = \infty,$$
$$\rho P''(\rho) + 2P'(\rho) < 0, \text{ for } \rho > 0.$$

Remark 1.1. By example if $f(\rho,w) = \rho$, then $g(\rho,w) = \rho w$. In this case we have the non-symmetric system with lineal damping.

Making $m = \rho w$, system (1) can be transformed in a symmetric system

$$\begin{cases} \rho_t + (\rho\phi(\rho,m)) = f(\rho,m), \\ m_t + (\rho w\phi(\rho,m))_x = g(\rho,m). \end{cases}$$

For this system, making $F(\rho,m) = (\rho\phi(\rho,m), m\phi(\rho,m))$, we obtain

$$dF = \begin{pmatrix} \phi + \rho\phi_\rho & \rho\phi_m \\ m\phi_\rho & \phi + m\phi_m \end{pmatrix},$$

so the eigenvalues and eigenvector of dF are given by

$$\lambda_1(\rho,m) = \Phi(\frac{m}{\rho}) - P(\rho), \qquad r_1 = (1, -\frac{\phi_\rho}{\phi_m}), \tag{2}$$

$$\lambda_2(\rho,m) = \Phi(\frac{m}{\rho}) - \rho P'(\rho), \quad r_2 = (1, \frac{m}{\rho}). \tag{3}$$

From (2) and (3), the $k-$Riemann invariants are given by

$$\begin{cases} W(\rho,m) = \Phi(\frac{m}{\rho}) - P(\rho), \\ Z(\rho,m) = \frac{m}{\rho}. \end{cases} \tag{4}$$

Moreover,

$$\nabla\lambda_1 \cdot r_1 = 0,$$
$$\nabla\lambda_2 \cdot r_2 = 2P'(\rho) + \rho P''(\rho).$$

By $\mathscr{C}_3$ condition, the system (6) is linear degenerate in the first characteristic field, non linear degenerate in the second characteristic field and non strictly hyperbolic. In this paper we obtain the main following theorem

Theorem 1.2. *If the initial data*

$$\rho(x,0) = \rho_0(x), w(x,0) = w_0(x) \in L^\infty(\mathbb{R}), \tag{5}$$

with $\rho_0(x) \geq 0$. The total variation of the Riemann invariants $W_0(x)$ be bounded, and the conditions $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ holds, then the Cauchy problem (1)*,* (5) *has a global bounded weak entropy solution and $w_x(x,t)$ is bounded in $L^1(\mathbb{R})$. Moreover, for w constant, ρ is the global weak solution of the scalar balance laws*

$$\rho_t + h(\rho)_x = f(\rho),$$

where $h(\rho) = \Phi(w)\rho - \rho P(\rho)$, $f(\rho) = f(\rho,w)$.

2 A priori bounds and existence

In order to get weak solutions, in this section we investigate the problem of the existence of the solutions for the parabolic regularization to the system (1)

$$\begin{cases} \rho_t + (\rho\phi(\rho,w)) = \varepsilon\rho_{xx} + f(\rho,w), \\ (\rho w)_t + (\rho w\phi(\rho,w))_x = \varepsilon(\rho w)_{xx} + g(\rho,w) \end{cases} \tag{6}$$

with initial data

$$\rho^\varepsilon(x,0) = \rho_0(x) + \varepsilon,\ w^\varepsilon(x,0) = w_0(x). \tag{7}$$

We consider the transformation $m = \rho w$, replacing in (6) we have

$$\begin{cases} \rho_t + (\rho\phi(\rho,m)) = \varepsilon\rho_{xx} + f(\rho,m), \\ m_t + (m\phi(\rho,m))_x = \varepsilon m_{xx} + g(\rho,m), \end{cases} \tag{8}$$

with initial data

$$\rho^\varepsilon(x,0), \quad m(x,0) = \frac{w_0(x)}{\rho^\varepsilon(x,0)}. \tag{9}$$

Proposition 2.1. *Let $\varepsilon > 0$. The Cauchy problem* (6)–(7) *has a unique solution for any (ρ_0, w_0). Moreover, if $(\rho_0, w_0) \in \Sigma$ their solutions $(\rho^\varepsilon, m^\varepsilon)$ satisfies*

$$0 < c \leq \rho(x,t) \leq M, \quad \left|\frac{m(x,t)}{\rho(x,t)}\right| \leq M.$$

The proof of this theorem is postponed at the end of the section. We begin with some lemmas that will be useful afterward.
Let $U = (\rho,m)^T$, $H(U) = (f(U), g(U))$ and $M = DF$ where $F(\rho,m) = (\rho\phi, m\phi)$. Then the system (8) can be written in the form

$$U_t = \varepsilon U_{xx} + MU_x + H.$$

For any C_1, C_2 constants let

$$G_1 = C_1 - W, \tag{10}$$
$$G_2 = Z - C_2, \tag{11}$$

where W, Z are the Riemann invariants given in (4).
We proof that the region

$$\Sigma = \{(\rho,m) : G_1 \leq 0, G_2 \leq 0\} \tag{12}$$

is an invariant region.

Lemma 2.2. *If $\rho \in C^{1,2}([0,T]\times\mathbb{R})$ satisfies*

$$\rho_t + (\rho\phi(w,\rho))_x = f(\rho,w),$$

with $\rho(0,\cdot) \geq 0$ and $w \in C^1(\mathbb{R})$ then $\rho(t,\cdot) \geq 0$. Moreover, if $\rho(0,\cdot) \geq \delta > 0$,

$$\int_0^T \int_{-\infty}^{\infty} \rho|w - w_0|dxdt < K$$

with K constant, then $\rho(x,t) \geq \delta(\varepsilon,T) > 0$ in $(0,T)$.

For the proof of this lemma see [11, Lemma 2.2].

Lemma 2.3. *The functions G_1 and G_2 defined in* (10) *and* (11)*, are quasi-convex.*

Proof. Let $r = (X,Y)$ be a vector. If $r\cdot\nabla G_1 = 0$ then $Y = X(\frac{m}{\rho} + \rho\frac{P'(\rho)}{\Phi'(w)})$ thus

$$\nabla^2 G_1(r,r) =$$
$$X^2\left(-\frac{1}{\rho}(2P''(\rho + \rho P'(\rho)) + \Phi''(w)(\frac{P'(\rho)}{\Phi'(w)})^2\right).$$

If $r\cdot\nabla G_2 = 0$ then $Y = \frac{m}{\rho}X$, thus we have

$$\nabla^2 G_2(r,r) = 0.$$

Lemma 2.4. *If the condition $\mathscr{C}_1$ holds, then G_1, G_2 satisfy*

$$\nabla G_1 \cdot H \leq 0,$$
$$\nabla G_2 \cdot H \leq 0.$$

Proof. From (10) and (11), we have that $\nabla G_1 = (-\Phi'(\frac{m}{\rho}) - P', \Phi'\frac{1}{\rho})$ and $\nabla G_2 = (-\Phi\frac{m}{\rho^2}, \Phi'\frac{1}{\rho})$ then

$$\nabla G_1 \cdot H = \frac{\Phi'}{\rho}(-\frac{m}{\rho}f + g) + P'f \leq 0,$$
$$\nabla G_2 \cdot H = \frac{\Phi'}{\rho}(-\frac{m}{\rho}f + g) \leq 0.$$

From the Theorem 14.7 in [12], the region Σ defined in (12) is an invariant region for the system (8). It follows from (10), (11) that

$$C_1 \leq \Phi(\frac{m}{\rho}) - P(\rho),$$
$$\frac{m}{\rho} \leq C_2.$$

Therefore,

$$C_1 - \Phi(C_2) \leq -P(\rho), \tag{13}$$

we appropriately choose C_1, C_2 such that

$$0 < \delta \leq \rho, \quad P(\rho) \leq \Phi(C_2) - C_1.$$

By (13) we have the proof of Proposition 2.1.

3 Weak convergence

In this section we show that the sequence $(\rho^{\varepsilon}, m^{\varepsilon})$ has a subsequence that converges the weak solutions to the system (8). For this we consider the following entropy-entropy flux pairs construct in [2] by the author

$$\eta(\rho, m) = \rho F(\frac{m}{\rho}),$$
$$q(\rho, m) = \rho F(\frac{m}{\rho})\phi(\rho, m).$$

The Hessian matrix of η is given by

$$\nabla^2\eta = \begin{pmatrix} F''\frac{m^2}{\rho^3} & -F''\frac{m}{\rho} \\ -F''\frac{m}{\rho} & F''\frac{1}{\rho} \end{pmatrix}$$

then we have that

$$\nabla^2\eta(X,X) = \frac{F''}{\rho}(\frac{m}{\rho}\rho_x - m_x)^2, \tag{14}$$

where $X = (\rho_x, m_x)$. If (η, q) is an entropy-entropy flux pair, multiplying in (8) by $\nabla\eta(\rho, m)$ we have

$$n_t + q_x = \varepsilon\eta_{xx} - \varepsilon\nabla^2\eta(X,X) + \nabla\eta \cdot G(\rho, m). \tag{15}$$

Replacing the equation (14) in (15) we have

$$n_t + q_x = \varepsilon\eta_{xx} - \varepsilon\frac{F''}{\rho}(\frac{m}{\rho}\rho_x - m_x)^2. + \nabla\eta \cdot G(\rho, m). \tag{16}$$

Chose a function $\varphi \in C_0^{\infty}(\mathbb{R}_+^2)$ satisfying $\varphi = 1$ on $[-L, L] \times [0, T]$. Multiplying (16) by φ and integrate the result in $(\mathbb{R}_+^2)$ we have

$$\begin{aligned}
&\int_{\mathbb{R}}(\eta\varphi)(x,T) - \int_{\mathbb{R}}(\eta\varphi)(x,0) \\
&- \int_0^T \int_{\mathbb{R}}(\eta\varphi_t + q\varphi_x)dxdt \\
&= -\varepsilon\int_0^T \int_{\mathbb{R}} \frac{F''}{\rho}(\frac{m}{\rho}\rho_x - m_x)^2 + \varepsilon\int_0^T \int_{\mathbb{R}} \eta\varphi_{xx}dxdt \\
&- \int_0^T \int_{\mathbb{R}} \nabla\eta \cdot G(\rho, m)dxdt.
\end{aligned}$$

From the Proposition 2.1 we have

$$\varepsilon\int_0^T \int_{-L}^{L} \frac{F''}{\rho}(\frac{m}{\rho}\rho_x - m_x)^2 dxdt \leq C. \tag{17}$$

As a consequence of the inequality (17) we have the following lemma.

Lemma 3.1. *For any $\varepsilon > 0$, if (ρ, m) is a solutions to the Cauchy problem* (8), (9)*, then $\sqrt{\varepsilon}\rho_x, \sqrt{\varepsilon}m_x$ are bounded in $L^2_{loc}(\mathbb{R}_+^2) \subset \mathscr{M}_{loc}$.*

We denote by $\mathscr{M}_{loc}$ the space of Radon Measures. For any bounded set $\Omega \subset \mathbb{R}_+^2$ we have

$$\begin{aligned}
&\|\varepsilon\eta_{xx}\|_{\mathrm{W}^{-1,2}(\Omega)} = \\
&\sqrt{\varepsilon}\|\eta\|_{\mathrm{L}^{\infty}(\Omega)}\|\sqrt{\varepsilon}u_x\|_{\mathrm{L}^2(\Omega)} \sup_{\varphi}\|\varphi_x\|_{\mathrm{L}^2} \to 0,
\end{aligned} \tag{18}$$

when $\varepsilon \to 0$.

Thereby,

$$\nabla\eta \cdot G(\rho, m) \in \mathrm{L}^{\infty}(\Omega) \subset \mathrm{L}^1(\Omega) \subset \mathscr{M}_{loc}. \tag{19}$$

Lemma 3.2.

$$g(\rho)_t + \left(\int^{\rho} g'(s)f'(s)ds + g(\rho)\Phi(w)\right)_x, \tag{20}$$

$$(\rho\Phi(w))_t + \big(\rho\Phi^2(w) + f(\rho)\Phi(w)\big)_x \tag{21}$$

are compact in $H^{-1}_{loc}(\mathbb{R}_+^2)$. Particularly

$$\rho_t + (\rho\Phi(w) - \rho P(\rho))_x$$

are compact in $H^{-1}_{loc}(\mathbb{R}_+^2)$.

Proof. The proof of (21) is a consequence of the Lemma 3.1 and the inequalities (18), (19) and the Murat's Lemma. Multiplying in (20) by g' we have

$$g(\rho)_t + \left(\int^{\rho} g'(s)f'(s)ds + g(\rho)\Phi(w)\right)_x =$$
$$g(\rho)_x x - \varepsilon g''(\rho)\rho_x^2 + \rho g'(\rho)\Phi(w)_x + g' f(\rho, w). \tag{22}$$

By a similar argument in the inequality (18) we have

$$\|\varepsilon g(\rho)_x x\|_{W^{-1,2}} \to 0, \text{as } \varepsilon \to 0.$$

From the Lemma 3.1 the last term in (22) is in $\mathscr{M}_{loc}$. By the Murat's Lemma we conclude the proof of (20).

Now we can stablish the proof of the main theorem. According to the Young's measures, there exists a probability measure $v_{x,t}$ associated with the bounded sequence $(\rho^\varepsilon, w^\varepsilon)$ such that for almost (x,t), $v_{x,t}$ satisfies the following Tartar equation,

$$\langle v_{x,t}, \eta_1 q_2 - \eta_2 q_1 \rangle = \langle v_{x,t}, \eta_1 q_2 \rangle - \langle v_{x,t}, \eta_2 q_1 \rangle$$

for any entropy-entropy flux pair, where $\langle v_{x,t}, f(\lambda)\rangle = \int_{\mathbb{R}^2} f(\lambda) dv_{x,t}(\lambda)$. We consider the following entropy-entropy flux pairs

$$\eta_1 = \rho^\varepsilon, \qquad q_1 = \rho^\varepsilon(\Phi(w\varepsilon) - P(\rho^\varepsilon)) + w^\varepsilon,$$
$$\eta_2 = \rho^\varepsilon w^\varepsilon, \quad q_2 = \rho^\varepsilon w^\varepsilon(\Phi(w\varepsilon) - P(\rho^\varepsilon)) + (w^\varepsilon)^2.$$

Notice that $\eta_1 q_2 - \eta_2 q_1 = 0$, then we have that

$$\overline{\eta_1 q_2} - \overline{\eta_2 q_1} = 0,$$

where $\overline{\varphi}$ denotes the weak-star limit $w* - \lim \varphi(u^\varepsilon)$. Therefore

$$\overline{\rho^\varepsilon}\,\overline{\rho^\varepsilon w^\varepsilon(\Phi(w\varepsilon) - P(\rho^\varepsilon)) + (w^\varepsilon)^2}$$
$$- \overline{\rho^\varepsilon w^\varepsilon}\,\overline{\rho^\varepsilon(\Phi(w\varepsilon) - P(\rho^\varepsilon)) + w^\varepsilon} = 0.$$

Using the the strong convergence of ρ^ε and $\Phi(w^\varepsilon)$ we have

$$\rho(\overline{(w^\varepsilon)^2} - (\overline{w^\varepsilon})^2) = 0. \tag{23}$$

The relation (23) includes the pointwise convergence of w^ε if $\rho > 0$.

4 Conclusion

This paper deals with a 2×2 inhomogeneous system of conservation laws of non-symmetric type, we extended the results of Lu [2] under adecuate conditions on the source term.

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