WELL-POSEDNESS TO THE CAUCHY PROBLEM ASSOCIATED TO THE NON LINEAL SCHRÖDINGER EQUATION

BUEN PLANTEAMIENTO DEL PROBLEMA DE CAUCHY ASOCIADO A LA ECUACIÓN NO LINEAL DE SCHRÖDINGER

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Abstract

In this paper we prove that the Cauchy problem associated to nonlinear Schrödinger equation is well-posed in Sobolev spaces H^s (R) using Parabolic regularization, Bona-Smith approximation and the methods proposed by Iorio in [7 - 9].

Keywords: NLS equation, well-posedness, parabolic regularization, Bona-Smith approximation. AMS subject classifications. 35Q55, 35G25

Resumen

En este artículo se muestra el buen planteamiento del problema de Cauchy asociado a la ecuación no lineal de Schrödinger en espacios de Sobolev H^s (R) usando el método de regularización parabólica,

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estimativas de Bona-Smith y los métodos propuestos por Iorio en [7 - 9].

Palabras clave: Ecuación NLS, buen planteamiento, regularización parabólica, estimativas de Bona-Smith.

Introduction

The linear Schrödinger equation was formulated by Erwin Schrödinger in 1925 and published in 1926. It describes how the quantum state of some physical system changes with time. The nonlinear Schrödinger equations are the main object of study of many physical problems.

In this work we will study the local well-posedness to the Cauchy problem associated to nonlinear Schrödinger equation (NLS):

$$\begin{cases} iv_t + \Delta v + |v|^{2\sigma}v = 0, & x \in \mathbb{R},\ t \geq 0, \\ v(0) = \phi. \end{cases} \qquad (1)$$

If $\sigma = 1$, the equation describes the propagation of a laser beam in a nonlinear optical medium whose index of refraction is proportional to the wave intensity. Also, the NLS equation successfully models other wave phenomena, such as water waves at the free surface of an ideal fluid as well as plasma waves.

Cao, Musslimani and Titi show the well-posedness of the following regularization to problem (1):

$$\begin{cases} iv_t + \Delta v + u|v|^{\sigma-1}v = 0, & x \in \mathbb{R},\ t \geq 0, \\ u - \alpha^2 \Delta u = |v|^{\sigma+1}, \\ v(0) = \phi, \end{cases} \qquad (2)$$

where $\alpha > 0$ and $\sigma \geq 1$.

This is called the Helmholtz-Schrödinger equation and was study in [3]. Note that when $\alpha = 0$, we get the NLS.

Well-Posedness to the Cauchy Problem
Associated to the Monlinear Schrödinger
Equation

Unlike the work of Cao, Musslimani and Titi, here we used Parabolic regularization method, Bona-Smith approximation and the methods proposed by Iorio [7, 9] to shows well-posedness of problem (1). The Parabolic regularization method consists of regularizing the equation (1) using the viscous term $-i\mu\mathcal{H}\Delta v$, constructing the solution to the Cauchy problem for nonlinear parabolic equation and taking the limit as the viscosity tends to zero, i.e., $\mu \to 0^+$. The Bona-Smith approximation were proposed in [2]. In this paper, the estimates are used to approximate the initial data using smooth functions and obtaining uniform bounds for the solutions. These techniques were also used in [1, 5].

We will use the following notation:

- $\mathbb{R}$ for the real numbers.
- $\widehat{\phi}$ for the Fourier transform of ϕ.
- If $s \in \mathbb{R}$, $H^s(\mathbb{R})$ is the Sobolev space with norm $\|\;\|_s$ and $(\,,\,)_s$ for its inner product.
- $B(X, Y)$ for the space of all continuous linear operator from X to Y, and $B(X)$ if $X = Y$.
- $C(I;X)$ for the space of all continuous functions on an interval I into Banach space X.
- $C^k(I; X)$ for the space of k times continuously differentiable functions on an interval I into Banach space X.
- $C_w(I; X)$ for the space of all weakly continuous functions on an interval I into Banach space X.

The Regularized Problem

We begin the analysis of the regularized problem

$$\begin{cases} iv_t + \Delta v + |v|^{2\sigma}v = -i\mu\mathcal{H}\Delta v, & x \in \mathbb{R},\, t \geq 0, \\ v(0) = \phi, \end{cases} \tag{3}$$

where $\mathcal{H}$ denotes the Hilbert transform

$$\mathcal{H}f(x) = \frac{1}{\pi}vp\int_{\mathbb{R}} \frac{f(y)}{x - y}dy. \tag{4}$$

Regularize the NLS equation using a viscous term $-i\mu\mathcal{H}\Delta v$ where $\mu > 0$.

We solved the regularized problem and then take the limit as viscosity tends to zero, i.e., $\mu \to 0+$. This procedure is known as Parabolic regularization or the Method of vanishing viscosity.

The Linear Equation

We consider the Cauchy problem associated to the linear part of equation (3)

$$\begin{cases} iv_t + \Delta v = -i\mu\mathcal{H}\Delta v, \\ v(0) = \phi. \end{cases} \tag{5}$$

Taking the Fourier transform in (5) we conclude that

$$v(x,t) = \left(e^{-i\xi^2(1+\mu sgn(\xi))t}\widehat{\phi}\right)^{\vee}.$$

For $t \geq 0$ define the linear operator

$$\mathbb{V}_\mu(t)\phi = \left(e^{-i\xi^2(1+\mu sgn(\xi))t}\widehat{\phi}\right)^{\vee} = e^{(i\Delta - \mu\mathcal{H}\Delta)t}\phi \tag{6}$$

Theorem 1. Let $\mu \geq 0$ be fixed. Then $V_\mu : [0,\infty) \to B(H^S(R))$ is a strongly continuous one-parameter unitary group in $H^S(R)$ for all $s \in R$.

Theorem 2. For all $\phi \in H^S(R)$, the function $v: [0,\infty) \to H^S(R)$ defined by $v(t) = V_\mu(t)\phi$, for all $t \in R$, is continuous with the time derivative computed in $(0,\infty)$ continuous. Moreover, v is the unique function in $C([0,\infty); H^S(R)) \cap C^1((0,\infty); H^S(R))$ solution of problem (5).
Note that, $v \in C((0,\infty); H^\infty(R)) \cap C^\infty((0,\infty); H^S(R))$.

Local Theory in $H^S(R)$, $s > 1/2$, $\mu > 0$

We use Banach's fixed theorem in a suitable function space, to find a local solution to the following integral equation associated to (3):

$$v(t) = \mathbb{V}_\mu(t)\phi + i\int_0^t \mathbb{V}_\mu(t-\tau)|v|^{2\sigma}v\,d\tau. \tag{7}$$

Well-Posedness to the Cauchy Problem
Associated to the Monlinear Schrödinger
Equation

Theorem 3. Suppose $s > \frac{1}{2}$. If $v \in C([0,T]; H^s(R)) \cap C^1((0,T]; H^{s-2}(R))$ is a solution of (3) then v is solution of the integral equation (7). Conversely, if $v \in C([0,T]; H^s(R))$ is a solution of (7) then $v \in C^1((0,T]; H^{s-2}(R))$ and satisfies (3).

Proposition 1. Let $F(v) = |v|^{2\sigma} v$ and $s > \frac{1}{2}$. Then

$$\|F(v) - F(w)\|_s \leq L(\|v\|_s, \|w\|_s) \|v - w\|_s \qquad (8)$$

where $L(\cdot, \cdot)$ is a continuous function, nondecreasing with respect to each of its arguments. In particular,

$$\|F(v)\|_s \leq L(\|v\|_s, 0) \|v\|_s. \qquad (9)$$

Proposition 2. Let $\phi, \varphi \in H^s(R)$ and $v, w \in C([0,T]; H^s(R))$ be two solutions of problem (3) satisfying $v(0) = \phi$ and $w(0) = \varphi$. If $s > \frac{1}{2}$ then

$$\|v_\mu(t) - w_\mu(t)\|_s \leq \|\phi - \varphi\|_s \exp\left(L(M_s, M_s) t\right) \qquad (10)$$

for all $t \in [0,T]$ and M_s is given by $M_s = \max\{\sup_{[0,T]} \|v(t)\|_s, \sup_{[0,T]} \|w(t)\|_s\}$.

In particular (3) has at most one solution.
Proof. Equation (7) implies

$$v(t) - w(t) = \mathbb{V}_\mu(t)(\phi - \varphi) + i \int_0^t \mathbb{V}_\mu(t - \tau)[|v(\tau)|^{2\sigma} v(\tau) - |w(\tau)|^{2\sigma} w(\tau)] \, d\tau \qquad (11)$$

and since $V_\mu(t)$ is unitary in $H^s(R)$ it follows that

$$\|v(t) - w(t)\|_s \leq \|\phi - \varphi\|_s + \int_0^t \left\| |v(\tau)|^{2\sigma} v(\tau) - |w(\tau)|^{2\sigma} w(\tau) \right\|_s d\tau$$

$$\leq \|\phi - \varphi\|_s + \int_0^t L(\|v(\tau)\|_s, \|w(\tau)\|_s) \|v(\tau) - w(\tau)\|_s \, d\tau$$

$$\leq \|\phi - \varphi\|_s + L(M_s, M_s) \int_0^t \|v(\tau) - w(\tau)\|_s \, d\tau$$

and Gronwall´s inequality implies the result.

Theorem 4. Let $s > \frac{1}{2}$ and $\phi \in H^s(\mathbb{R})$. Then there exist a $T = T(\|\phi\|_s, M) > 0$ and a unique $v \in C([0, T]; H^s(\mathbb{R}))$ solution of problem (3).

Proof. Let $M, T > 0$. T will be conveniently chosen later. Consider

$$(\mathcal{A}v)(t) = \mathbb{V}_\mu(t)\phi + \int_0^t \mathbb{V}_\mu(t - \tau)|v|^{2\sigma} v(\tau)\, d\tau, \quad v \in \mathcal{X}(M, T, \phi) \tag{12}$$

defined on the complete metric space

$$\mathcal{X}(M, T, \phi) = \{v \in C([0, T]; H^s(\mathbb{R})) : \|v(t) - \mathbb{V}_\mu(t)\phi\|_s \leq M\} \tag{13}$$

provided with the distance $d(u, v) = \sup\limits_{t \in [0, T]} \|u(t) - v(t)\|_s$.

Is easy to show that the map $\mathcal{A}$ has a unique fixed point in the complete metric space $(\mathcal{X}(M, T, \phi), d)$.

Theorem 5. The map $\phi \mapsto v$ defined by (3) is continuous. More precisely: let $\phi^{(n)} \in H^s(\mathbb{R})$, $n = 1, 2, 3, \ldots, \infty$ be such that $\phi^{(n)} \to \phi^{(\infty)}$ in $H^s(\mathbb{R})$ when $n \to \infty$. Let $v^{(n)} \in C([0, T_n]; H^s(\mathbb{R})) \cap C^1((0, T_n]; H^{s-2}(\mathbb{R}))$, where $T_n = T(M, \|\phi^{(n)}\|_s)$, be the solutions of (3) with $v^{(n)}(0) = \phi^{(n)}$, $n = 1, 2, 3, \ldots, \infty$. Let $\mathrm{T} \in (0, T_\infty)$. Then the solutions $v^{(n)}$ are defined in $[0, T]$ for all sufficiently large and

$$\lim_{n \to \infty} \sup_{[0, T]} \left\| v^{(n)} - v^{(\infty)} \right\|_s = 0.$$

Theorem 6. Problem (3) is locally well-posed in the sense of Hadamard.

Local Well-posedness. Case $\mu = 0$

Proposition 3. Let $\phi \in H^s(\mathbb{R})$, $s > \frac{1}{2}$. Then there exist a $T(\|\phi\|_s) > 0$ and a function $\rho \in C([0, T]; [0, \infty))$, both independent of μ, such that $T \leq T^*(\mu, \phi)$ (the time of maximal interval of existence v_μ *in* H^s) for all $\mu > 0$ and

$$\|v_\mu(t)\|_s \leq \rho(t) \quad \text{for all } t \in [0, T]. \tag{14}$$

Proof.

$$\frac{1}{2}\partial_t \|v_\mu(t)\|_s^2 = (v, i\Delta v)_s + (v, i|v|^{2\sigma}v)_s - \mu(v, H\Delta v)_s$$

$$\leq \|v_\mu(t)\|_s^{2(\sigma+1)}$$

Let $\rho \in C([0, T^*); [0, \infty))$ be the maximally extended solution of

$$\begin{cases} \frac{\rho(t)}{dt} = 2C_s\, [\rho(t)]^{\sigma+1} \\ \rho(0) = \|\phi\|_s^2. \end{cases} \tag{15}$$

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Well-Posedness to the Cauchy Problem
Associated to the Monlinear Schrödinger
Equation

For comparision theorem of the theory of ODEs, we must have

$$\|v_\mu(t)\|_s^2 \leq \rho(t).$$

Since $\rho(t)$ and T^* do not depend on μ, the usual extension method shows that we must have $0 < T(\|\phi\|_s) \leq T^*$ for all $\mu > 0$. This finishes the proof.

Theorem 7. Let $s > \frac{1}{2}$ be fixed. Then, for every $\phi \in H^s(\mathbb{R})$, there exis a $T = T(\|\phi\|_s) > 0$ and a function $v_0 \in C_w([0,T]; H^s(\mathbb{R})) \cap C_w^1([0,T]; H^{s-2}(\mathbb{R}))$ such that $v_0(0) = \phi$ and v_0 is a solution of equation (3) in the weak sense, that is,

$$\partial_t(v_0(t), \varphi)_{s-2} = (i\Delta v_0(t) + i|v_0(t)|^{2\sigma}v_0(t) - \mathcal{H}\Delta v_0(t), \varphi)_{s-2} \tag{16}$$

for all $\varphi \in H^{s-2}(\mathbb{R})$ and $t \in [0,T]$.
Moreover $\|v_0(t)\|_s \leq \rho(t)$ for all $t \in [0,T]$ where ρ is as in Proposition 3.

Proof. Let $T(\|\phi\|_s)$ be as in Proposition 3. We will show that v_μ converges to a function $v_0 \in C([0, T(\|\phi\|_s)]; L^2(\mathbb{R}))$ in the L^2–norm uniformly over $[0, T(\|\phi\|_s)]$. Let v_{μ_1}, v_{μ_2} be solutions of (3) with $v_{\mu_1}(0) = \phi = v_{\mu_2}(0)$, then

$$\frac{1}{2}\partial_t\|v_{\mu_1} - v_{\mu_2}\|_0^2 \leq -(\mu_1 - \mu_2)\int_\mathbb{R}(v_{\mu_1} - v_{\mu_2})\mathcal{H}\Delta v_{\mu_2}\, dx$$
$$+ i\int_\mathbb{R}(v_{\mu_1} - v_{\mu_2})\left(|v_{\mu_1}|^{2\sigma}v_{\mu_1} - |v_{\mu_2}|^{2\sigma}v_{\mu_2}\right)\, dx$$

Therefore,

$$|v_{\mu_1}|^2 - |v_{\mu_2}|^2 = v_{\mu_1}(\overline{v}_{\mu_1} - \overline{v}_{\mu_2})Q(|v_{\mu_1}|, |v_{\mu_2}|) + \overline{v}_{\mu_2}(v_{\mu_1} - v_{\mu_2})Q(|v_{\mu_1}|, |v_{\mu_2}|)$$

where

$$Q(x,y) = x^{2(\sigma-1)} + x^{2(\sigma-2)}y^2 + \cdots + x^2 y^{2(\sigma-2)} + y^{2(\sigma-1)}.$$

Let $M = \sup\limits_{t\in[0,T(\|\phi\|_s)]}\sqrt{\rho(t)}$ where ρ is the function defined in the proof of Proposition 3. Then

$$\frac{1}{2}\partial_t\|v_{\mu_1} - v_{\mu_2}\|_0^2 \leq |\mu_1 - \mu_2|\|\partial_x(v_{\mu_1} - v_{\mu_2})\|_0\|\partial_x u_{\mu_2}\|_0$$
$$+ \|v_{\mu_1} - v_{\mu_2}\|_0^2 \||v_{\mu_1}|^{2\sigma} + (|v_{\mu_1}||v_{\mu_2}| + |v_{\mu_2}|^2)Q(|v_{\mu_1}|, |v_{\mu_2}|)\|_{L^\infty}$$
$$\leq M^2|\mu_1 - \mu_2| + CM^\sigma\|v_{\mu_1} - v_{\mu_2}\|_0^2$$

The Gronwall's inequality shows that exists a constant $\widetilde{C} > 0$

$$\|v_{\mu_1} - v_{\mu_2}\|_0^2 \leq \widetilde{C}|\mu_1 - \mu_2| \quad \text{for all } t \in [0, T(\|\phi\|_s)]. \tag{17}$$

Since L^2 is complete, then there exists $v_0 \in C([0,T]; L^2)$ such that

$$\lim_{\mu\to 0}\sup_{t\in[0,T(\|\phi\|_s)]}\|v_\mu(t) - v_0(t)\|_0 = 0.$$

It follows that v_μ converges weakly to v_0 in $H^s(\mathbb{R})$ uniformly over $[0, T(\|\phi\|_s)]$. In particular $v_0(t)$ is weakly continuous and uniformly bounded by the function $\sqrt{\rho(t)}$.

Since $v_\mu \rightharpoonup v_0$ in $H^s(\mathbb{R})$ it follows that

$$\Delta v_\mu \rightharpoonup \Delta v_0 \text{ in } H^{s-2}(\mathbb{R})$$

$$\mathcal{H}\Delta v_\mu \rightharpoonup \mathcal{H}\Delta v_0 \text{ in } H^{s-2}(\mathbb{R})$$

$$|v_\mu|^{2\sigma} v_\mu \rightharpoonup |v_0|^{2\sigma} v_0 \text{ in } H^s(\mathbb{R})$$

uniformly over $[0, T(\|\phi\|_s)]$, where $\rightharpoonup$ stands for weak convergence.

Taking the limit as $\mu \to 0+$ we obtain

$$\partial_t(v_0(t), \varphi)_{s-2} = (i\Delta v_0(t) + i|v_0(t)|^{2\sigma} v_0(t) - \mathcal{H}\Delta v_0(t), \varphi)_{s-2}.$$

for all $t \in [0, T(\|\phi\|_s)]$.

Corollary 1. Let v_0 be as in preceding Theorem. Then $v_0 \in AC([0, T]; H^{s-2})$.

AC $(I; X)$ denotes the collection of all absolutely continuous functions from the interval I into the Banach space X, that is, the set of all functions $F : I \to X$ of the form

$$F(x) = F(a) + \int\limits_a^x G(t)\, dt$$

where $G : I \to X$ is Bochner integrable.

Proposition 4. Let $T > 0$ be fixed, $\phi_j \in H^s(\mathbb{R})$, $j = 1, 2$, and let $v_j : [0, T] \to H^s(\mathbb{R})$ be bounded functions such that $v_j(0) = \phi_j$ and

$$v_j \in C([0, T]; L^2(\mathbb{R})) \cap C_w([0, T]; H^s(\mathbb{R})) \cap AC([0, T]; H^{s-2}(\mathbb{R}))$$

Then

$$\|v_1(t) - v_2(t)\|_0 \le \|\phi_1 - \phi_2\|_0\, e^{\widetilde{G}(M,M)t}$$

where $M = \max\left\{ \sup_{[0,T]} \|v_1(t)\|_s, \sup_{[0,T]} \|v_2(t)\|_s \right\}$.

Theorem 8. Let v_0 be as in Teorem 7. Then $v_0 \in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-2}(\mathbb{R}))$ and is unique solution of NLS.

Prof. The uniqueness follows form Proposition 4. Since $\|v_0(t)\|_s \le \sqrt{\rho(t)}$ it follows at once that

$$\liminf_{t \to 0+} \|v_0(t)\|_s = \limsup_{t \to 0+} \|v_0(t)\|_s = \|\phi\|_s,$$

so that the limit of $\|v_0(t)\|_s$ exists as $t \to 0+$ and

$$\lim_{t \to 0+} \|v_0(t)\|_s = \|\phi\|_s.$$

Well-Posedness to the Cauchy Problem
Associated to the Monlinear Schrödinger
Equation

Left continuity follows from the invariance of equation in (3) under the transformation

$$(x,t) \mapsto (-x, t' - t).$$

Lemma 1 (Bona-Smith approximation). Let $\phi \in H^s(\mathbb{R})$, $s > \frac{1}{2}$, $\tau \geq 0$ and introduce

$$\phi^\tau = \exp\left(-\tau\left(1-\partial_x^2\right)^{\frac{s}{2}}\right)\phi = \left(\widehat{\phi}\exp\left(-\tau\left(1+|\cdot|^2\right)^{\frac{s}{2}}\right)\right)^{\vee}. \qquad (18)$$

Then

$$\lim_{\tau \to 0+} \|\phi^\tau - \phi\|_s = 0 \qquad (19)$$

and, there exist a constant $C = C(s)$ such that

$$\|\phi^\tau\|_{s+1} \leq C \left[1 + \left(\frac{1}{s\tau}\right)^{\frac{2}{s}}\right]^{\frac{1}{2}} \|\phi\|_s \qquad (20)$$

and

$$\left\|\phi^\tau - \phi^\theta\right\|_0 \leq C|\tau - \theta| \|\phi\|_s . \qquad (21)$$

Proposition 5. Let $s > \frac{1}{2}$, $\phi \in H^s(\mathbb{R})$, ϕ^τ be as in preceding Lemma. If v_0^τ is solution of NLS with $v_0^\tau = \phi^\tau$ for all $\tau > 0$, then there are constants $C = C(s, \|\phi\|_s, T) > 0$ and $\eta = \eta(s) \in (0,1)$ such that

$$\left\|v_0^\tau - v_0^\theta\right\|_s^2 \leq C \left(\left\|\phi^\tau - \phi^\theta\right\|_s^2 + \tau^{1-\eta}\right) \qquad (22)$$

for sufficiently small and $0 \leq \theta \leq \tau$.

Preposition 6. The map $\phi \mapsto v_0$ defined of H^s to $C([0,T]; H^s)$ is continuous.

Global theory in $H^s(\mathbb{R})$

For Global existence we study the Hamilitonian structure with small initial data. The global well-posedness of the Cauchy problem associated to

$$\begin{cases} iv_t + \Delta v + u|v|^{\sigma-1}v = 0, & x \in \mathbb{R}, \, t \geq 0, \\ u - \alpha^2\Delta u = |v|^{\sigma+1}, \\ v(0) = \phi, \end{cases}$$

with $\alpha > 0$ was study in [3]:

Theorem 9. Let $\phi \in H^1(\mathbb{R})$. If $1 \leq \sigma < 3$ then there esists a unique solution $v \in C([0, \infty); H^1(\mathbb{R}))$.

Furthermore, the charge

$$Q(v) = \int_{\mathbb{R}} |v(x,t)|^2 \, dx = \|v\|_0^2 \tag{23}$$

and Hamiltonian

$$H(v) = \int_{\mathbb{R}} \left(|\nabla v(x,t)|^2 - \frac{u(x,t)|v(x,t)|^{\sigma+1}}{\sigma+1} \right) dx \tag{24}$$

are conserved in time.

The followin result requires no smallness condition for initial data.

Theorem 10. If $1 \leq \sigma < 2$ then the Cauchy problem associated to NLS with initial date $\phi \in H^1(\mathbb{R})$ is global well-posed in $H^1(\mathbb{R})$.

We used that the charge

$$Q(v) = \int_{\mathbb{R}} |v(x,t)|^2 \, dx = \|v\|_0^2 \tag{25}$$

and Hamiltonian

$$H(v) = \int_{\mathbb{R}} \left(|\nabla v(x,t)|^2 - \frac{|v(x,t)|^{2\sigma+2}}{2\sigma+2} \right) dx \tag{26}$$

are conserved in time for show global well-posedness [9].

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Well-Posedness to the Cauchy Problem
Associated to the Monlinear Schrödinger
Equation

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