

Hopf algebras and skew PBW extensions

Álgebras de Hopf y extensiones PBW torcidas

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Abstract

In this article we relate some Hopf algebra structures over Ore extensions and over skew PBW extensions of a Hopf algebra. These relations are illustrated with examples. We also show that Hopf Ore extensions and generalized Hopf Ore extensions are Hopf skew PBW extensions.

Key words: Hopf algebras, skew PBW extensions, Hopf Ore extensions, generalized Hopf Ore extensions.

Resumen

En este artículo se relacionan algunas estructuras de álgebra de Hopf sobre extensiones de Ore y extensiones PBW torcidas de un álgebra de Hopf. Estas relaciones son ilustradas con ejemplos. También se demuestra que las extensiones Hopf Ore y las extensiones Hopf Ore generalizadas son extensiones PBW torcidas de Hopf.

Palabras clave: Álgebras de Hopf, extensiones PBW torcidas, extensiones de Ore, extensiones Hopf Ore generalizadas.

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1. INTRODUCTION

Hopf algebras are an active field of research in recent years. They have relations with noncommutative geometry. If R is a Hopf algebra it is natural to ask under what conditions the Hopf structure of R can be extended to an Ore extension $B = R[x; \sigma, \delta]$ of R . Panov in [10] answered the above question under the additional hypothesis that x is a skew primitive element of B , that is, there are group-like elements $a, b \in R$ such that $\Delta(x) = a \otimes x + x \otimes b$, where Δ denotes the coproduct of B . Brown, O’Hagan, J. Zhang and Zhuang in [2] investigated when an Ore extension $B = R[x; \sigma, \delta]$ of a Hopf algebra R admits a Hopf algebra structure, generalizing a the result of Panov, the main result in this direction is Theorem 2.4. They also studied the structure of Hopf algebra for iterated Ore extensions and obtained a large family of connected Hopf algebras of finite Gelfand-Kirillov dimension (or GK-dimension for short), including for example all enveloping algebras of finite dimensional solvable Lie algebras.

In the year 2018, You, Wang and Chen in [24] investigated some Hopf algebra structures over an Ore extension of a Hopf algebra. They gave necessary and sufficient conditions for a certain type of Ore extension of a Hopf algebra to have a Hopf algebra structure. For this purpose they defined the generalized Hopf Ore extensions.

On the other hand, Gallego and Lezama in [3] defined skew PBW extensions. Many properties of these extensions have been studied recently (see for example [5, 6, 9, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23]). Some algebras, such as the Ore extensions, the enveloping algebra of a finite dimensional Lie algebra and the quantum plane are skew PBW extensions (see [3]). The Hopf algebra structure of these algebras was studied in [2, 10, 24]. In the literature does not explicitly find a structure of Hopf algebra for skew PBW extensions. Therefore in this paper we study the Hopf algebra structure in some skew PBW extensions and their relation to Hopf Ore extensions defined in [10], Hopf Ore extensions and iterated Hopf Ore extensions defined in [2], and generalized Hopf Ore extensions defined in [24].

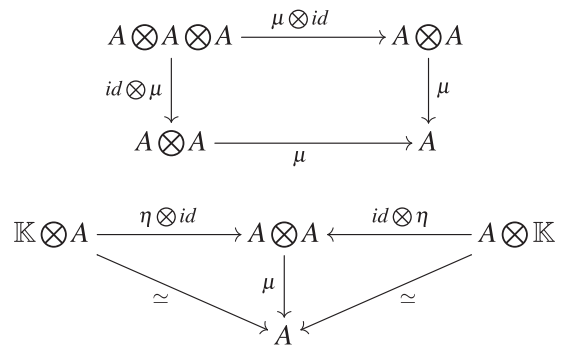
In Section 2 we present some definitions and properties related to Hopf algebras, Ore extensions and skew PBW extensions. In this section defined Hopf skew PBW extensions and present some examples and counterexamples of these algebras. We also show that pre-commutative skew PBW extensions of \mathbb{K} , with $c_{ij} = 1$ is a cocommutative Hopf algebra (Theorem 21). In Section 3 we relate P-Hopf Ore extensions, Hopf Ore extensions and iterated Hopf Ore extensions, generalized Hopf Ore extensions and Hopf skew PBW extensions. Examples and counterexamples of these relations are presented. The main results of this section are Remark 30, Remark 32, Proposition 33, Theorem 40, Remark 41 and Theorem 45.

2. HOPF SKEW PBW EXTENSIONS

We establish the following notation: the symbol \mathbb{N} is used to denote the set of natural numbers including zero, the letter \mathbb{K} denotes a field, every algebra is a \mathbb{K} -algebra and vector spaces are vector spaces over \mathbb{K} .

2.1 Hopf algebras

Definition 1. An *algebra* is a vector space A together with two linear maps, a multiplication $\mu: A \otimes A \rightarrow A$, and unit map $\eta: \mathbb{K} \rightarrow A$ such that the following diagrams commute:



where the lower left and right maps are simply scalar multiplication.

Definition 2. If A and B are algebras with respective multiplications μ_A and μ_B then:

a linear map $g: A \rightarrow B$ is an *algebra morphism* if $g \circ \mu_A = \mu_B \circ (g \otimes g)$ i.e. the following diagram commutes.

$$\begin{array}{ccc}
 A \otimes A & \xrightarrow{\mu_A} & A \\
 g \otimes g \downarrow & & \downarrow g \\
 B \otimes B & \xrightarrow{\mu_B} & B
 \end{array}$$

That is, if $a, b \in A$ then

$$g(ab) = g(a)g(b).$$

Given vector spaces V and W , we define the *flip map* as the map $\tau : V \otimes W \rightarrow W \otimes V$ such that $\tau(v \otimes w) = w \otimes v$ for all $v \in V, w \in W$.

Definition 3. A *coalgebra* is a vector space C together with two linear maps, comultiplication $\Delta : C \rightarrow C \otimes C$ and counit $\varepsilon : C \rightarrow \mathbb{K}$, such that the following two diagrams commute:

$$\begin{array}{ccccc}
 C & \xrightarrow{\Delta} & C \otimes C & & \\
 \Delta \downarrow & & \downarrow \Delta \otimes id & & \\
 C \otimes C & \xrightarrow{id \otimes \Delta} & C \otimes C \otimes C & & \\
 \cong \swarrow & & \cong \searrow & & \\
 \mathbb{K} \otimes C & \xleftarrow{\varepsilon \otimes id} & C & \xrightarrow{id \otimes \varepsilon} & C \otimes \mathbb{K} \\
 & & \Delta \downarrow & & \\
 & & C \otimes C & &
 \end{array}$$

Let C be a coalgebra. We say that C is *cocommutative* if $\tau\Delta = \Delta$, where τ is the flip map (see [8]).

Definition 4. If $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ are coalgebras. A *coalgebra morphism* is a linear map $\varphi : C \rightarrow D$ such that

$$\Delta_D \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_C \quad \varepsilon_D \otimes \varphi = \varepsilon_C$$

i.e. the following diagram commutes.

$$\begin{array}{ccc}
 C & \xrightarrow{\varphi} & D \\
 \Delta_C \downarrow & & \downarrow \Delta_D \\
 C \otimes C & \xrightarrow{\varphi \otimes \varphi} & D \otimes D
 \end{array}$$

Definition 5. Let C be any coalgebra, and let $c \in C, c \neq 0, c$ is *group-like* if $\Delta(c) = c \otimes c$ and $\varepsilon(c) = 1$. The set of group-like elements in C is denoted by $G(C)$. A *primitive element* $c \in G(C)$ satisfies $\Delta(c) = c \otimes 1 + 1 \otimes c$. Let $P(C)$ denote the set of all primitive elements of C .

Note that $P(C)$ is a Lie algebra under the commutator difference product $(x, y) \mapsto [x, y] = xy - yx$ (see [7]). Indeed: let $x, y \in P(C)$, then

$$\begin{aligned}
 \Delta(xy - yx) &= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) \\
 &\quad - (y \otimes 1 + 1 \otimes y)(x \otimes 1 + 1 \otimes x) \\
 &= xy \otimes 1 + x \otimes y + y \otimes x + 1 \otimes xy \\
 &\quad - yx \otimes 1 - y \otimes x - x \otimes y - 1 \otimes yx \\
 &= (xy - yx) \otimes 1 + 1 \otimes (xy - yx)
 \end{aligned}$$

Definition 6. Given a space B, B is a *bialgebra* if (B, Δ, ε) is a coalgebra, (B, μ, η) is an algebra and either of the following equivalent conditions is true:

- (i) Δ and ε are algebra morphisms.
- (ii) μ and η are coalgebra morphisms.

This bialgebra structure is often denoted by $(B, \mu, \eta, \Delta, \varepsilon)$.

Definition 7. Given an algebra (A, μ, η) , a coalgebra (C, Δ, ε) and two linear maps $f, g : C \rightarrow A$ then the *convolution* of f and g is the linear map $f * g : C \rightarrow A$ defined by

$$(f * g)(c) = \mu \circ (f \otimes g) \circ \Delta(c), \quad c \in C.$$

Definition 8. Let $(B, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. An endomorphism S of B is called *antipode* for the bialgebra if

$$id_B * S = S * id_B = \eta \circ \varepsilon.$$

In Sweedler's notation:

$$\sum S(h_1)h_2 = \varepsilon(h)1_B = \sum h_1S(h_2), \quad \forall h \in B.$$

The group-like elements form a group with inverse given by the antipode. By the definition of an antipode, if g is group-like then $S(g)$ satisfies

$$S(g)g = 1, \quad gS(g) = 1.$$

Definition 9. A *Hopf algebra* is a bialgebra with an antipode. A Hopf algebra is said to be *connected* if its unique simple subcoalgebra is just the base field \mathbb{K} . The *coradical* of a Hopf algebra is defined as the sum of its simple subcoalgebras.

Zhuang in [26, Theorem 6.6] proved that if the base field \mathbb{K} is algebraically closed of

characteristic 0 and H is a connected Hopf algebra, then H is a domain.

Let $Alg_{\mathbb{K}}(H, \mathbb{K})$ be the \mathbb{K} -vector space of unital algebra homomorphisms from H to \mathbb{K} , for $\alpha \in Alg_{\mathbb{K}}(H, \mathbb{K})$, the *left winding automorphisms* τ'_α is the algebra endomorphism $\mu \circ (\alpha \otimes Id) \circ \Delta : H \rightarrow H$, i.e., in Sweedler's notation:

$$\tau'_\alpha(h) = \sum \alpha(h_1)h_2, \quad (1)$$

for all $h \in H$. The *right winding automorphisms* τ''_α is the map $\mu \circ (Id \otimes \alpha) \circ \Delta : H \rightarrow H$, or in Sweedler's notation:

$$\tau''_\alpha(h) = \sum h_1\alpha(h_2), \quad (2)$$

for all $h \in H$. If B is an algebra, we call an algebra homomorphism $\chi : B \rightarrow \mathbb{K}$ a *character* of B , and its kernel a *character ideal*.

In what follows, for a Hopf algebra H , we always denote the comultiplication by Δ , the counit by ε , the antipode by S and for $h \in H$, write $\Delta(h) = h_1 \otimes h_2$.

2.2 Ore extensions and skew PBW extensions

Definition 10. If R is an algebra and σ is an algebra automorphism of R , then a σ -*derivation* δ of R is a \mathbb{K} -linear endomorphism of R such that

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b \quad (3)$$

for all $a, b \in R$. The *Ore extension* $B = R[x; \sigma, \delta]$ is the algebra generated by R and x , subject to the relations

$$xr = \sigma(r)x + \delta(r), \quad (4)$$

for all $r \in R$.

Let $B = R[x_1; \sigma_1, \delta_1]$ be an Ore extension of R and $C = B[x_2; \sigma_2, \delta_2]$ be an Ore extension of B , then C is a free left R -module with a basis $\{x_1^{n_1} x_2^{n_2}, n_1, n_2 \geq 0\}$. The ring C is called an iterated Ore extension of R . More generally we can consider the *iterated Ore extension* $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ where σ_i, δ_i are defined on $R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}]$.

Now, we recall some definitions and elementary properties of skew PBW extensions.

Skew PBW extensions are a generalization of PBW extensions. *PBW extensions* were defined by Bell and Goodearl in [1].

Definition 11. Let R and A be rings. It is said that A is a *Poincaré-Birkhoff-Witt extension* of R , noted PBW, if the following conditions hold:

- (i) $R \subseteq A$.
- (ii) There exist finitely many elements $x_1, \dots, x_n \in A$ such that A is a left R -free module with basis

$$Mon(A) := Mon\{x_1, \dots, x_n\} := \{x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

In this case it is also said that A is a *ring of a left polynomial type over R* with respect to $\{x_1, \dots, x_n\}$ and $Mon(A)$ is the set of standard monomials of A . Moreover, $x_1^0 \dots x_n^0 := 1 \in Mon(A)$.

- (iii) $x_i r - r x_i \in R$, for each $r \in R$ and $1 \leq i \leq n$.
- (iv) $x_i x_j - x_j x_i \in R + R x_1 + \dots + R x_n$, for any $1 \leq i, j \leq n$.

Definition 12. Let R and A be algebras. We say that A is a *skew PBW extension of R* if the following conditions hold:

- (i) $R \subseteq A$;
- (ii) there exist finitely many elements $x_1, \dots, x_n \in A$ such that A is a left free R -module, with basis the set of standard monomials

$$Mon(A) := \{x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

Moreover, $x_1^0 \cdots x_n^0 := 1 \in Mon(A)$.

- (iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \quad (5)$$

- (iv) For $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_{i,j} x_j \in R + R x_1 + \dots + R x_n. \quad (6)$$

Under these conditions, we will write $A = \sigma(R)(x_1, \dots, x_n)$. If A is a skew PBW extension with a structure of Hopf algebra, we say that A is a *Hopf skew PBW extension*.

Note that Poincaré-Birkhoff-Witt extensions, Ore extensions, the universal enveloping algebra of a finite dimensional Lie algebra and commutative polynomial algebras are skew PBW extensions. For more examples, details and to check other recent properties related to skew PBW extensions, see [3, 5, 9, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23].

2.3 Examples

Throughout, \mathbb{K} is an algebraically closed field of characteristic 0. From the examples that we present below, Examples 13, 14, 15 and 16 are Hopf skew PBW extensions. Example 17 is a skew PBW extension but this has no structure of Hopf algebra. Examples 18 and 19 are not Hopf skew PBW extensions, since these have a structure of Hopf algebra but are not skew PBW extensions.

Example 13. Let $\mathbb{K}[x]$ the *Polynomial algebra*. We define the coalgebra structure by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0.$$

$\mathbb{K}[x]$ is an example of cocommutative coalgebra. An antipode is define by

$$S(x) = -x$$

The property of the antipode for x , in $\mathbb{K}[x]$ bialgebra, is given by

$$1S(x) + xS(1) = S(1)x + S(x)1 = 0 = \varepsilon(x)1.$$

With the above definitions $\mathbb{K}[x]$ is a Hopf algebra. Moreover, $\mathbb{K}[x]$ is a skew PBW extension of \mathbb{K} . Thus, $\mathbb{K}[x]$ is a Hopf skew PBW extension.

Example 14. *Universal enveloping algebra of a Lie algebra.* Let \mathfrak{g} be a finite dimensional Lie algebra with basis $\{x_1, \dots, x_n\}$; the universal enveloping algebra of \mathfrak{g} , $U(\mathfrak{g})$, is a skew PBW extension of \mathbb{K} , where $x_i r - r x_i = 0$ and $x_i x_j - x_j x_i = [x_i, x_j] \in \mathfrak{g} = \mathbb{K} + \mathbb{K}x_1 + \dots + \mathbb{K}x_n$, for any $r \in \mathbb{K}$ and $1 \leq i, j \leq n$. $U(\mathfrak{g})$, is a Hopf algebra with

comultiplication $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$ for $1 \leq i \leq n$. This rule can be uniquely extended to all $U(\mathfrak{g})$ (extending algebraically or antialgebraically). The counit ε is given by $\varepsilon(x_i) = 0$, for all i and extended to $U(\mathfrak{g})$. The antipode S is given by $S(x_i) = -x_i$. $U(\mathfrak{g})$ is commutative if and only if \mathfrak{g} is abelian. $U(\mathfrak{g})$ is cocommutative, since $\tau\Delta(x_i) = \tau(x_i \otimes 1 + 1 \otimes x_i) = 1 \otimes x_i + x_i \otimes 1 = \Delta(x_i)$ for $1 \leq i \leq n$. $U(\mathfrak{g})$ is a connected Hopf algebra and $P(U(\mathfrak{g})) = \mathfrak{g}$.

Example 15. Let $A(\lambda_1, \lambda_2, \alpha)$ be the algebra generated by elements x, y, z satisfying the following relations,

$$\begin{aligned} xy - yx &= 0, \\ zx - xz &= \lambda_1 x + \alpha y, \\ zy - yz &= \lambda_2 y, \end{aligned} \quad (7)$$

where $\alpha = 0$ if $\lambda_1 \neq \lambda_2$ and $\alpha = 0$ or 1 if $\lambda_1 = \lambda_2$. From relations (7) we have that $A(\lambda_1, \lambda_2, \alpha)$ is a skew PBW extension of $\mathbb{K}[x, y]$, i.e., $A(\lambda_1, \lambda_2, \alpha) = \sigma(\mathbb{K}[x, y])(z)$. Also, $A(\lambda_1, \lambda_2, \alpha) = \mathbb{K}[x, y][z; \sigma, \delta]$, where $\sigma(x) = x$, $\sigma(y) = y$, $\delta(x) = \lambda_1 x + \alpha y$ and $\delta(y) = \lambda_2 y$. From [26, Example 7.1] we have that $A(\lambda_1, \lambda_2, \alpha)$ becomes a Hopf algebra via

$$\begin{aligned} \Delta(x) &= 1 \otimes x + x \otimes 1, & \varepsilon(x) &= 0, \\ \Delta(y) &= 1 \otimes y + y \otimes 1, & \varepsilon(y) &= 0, \\ \Delta(z) &= 1 \otimes z + x \otimes y + z \otimes 1, & \varepsilon(z) &= 0, \\ S(x) &= -x, & S(y) &= -y, & S(z) &= -z + xy. \end{aligned}$$

Therefore $A(\lambda_1, \lambda_2, \alpha)$ is a Hopf skew PBW extension. Moreover $A(\lambda_1, \lambda_2, \alpha)$ is a connected Hopf algebra of GK-dimension three. (see [26, Proposition 7.3]).

Example 16. Let $B(\lambda)$ be the algebra generated by elements x, y, z satisfying the following relations,

$$\begin{aligned} xy - yx &= y, \\ zx - xz &= -z + \lambda y, \\ zy - yz &= \frac{1}{2}y^2, \end{aligned} \quad (8)$$

where $\lambda \in \mathbb{K}$. From relations (8) we have that $B(\lambda)$ is a skew PBW extension of $\mathbb{K}[y]$, i.e., $B(\lambda) = \sigma(\mathbb{K}[y])(x, z)$. Also, $B(\lambda) = \mathbb{K}[y][x; \sigma_1, \delta_1][z; \sigma_2, \delta_2]$, where $\sigma_1(y) = y$, $\delta_1(y) = y$, $\sigma_2(y) = y$, $\delta_2(y) = \frac{1}{2}y^2$, $\sigma_2(x) = x$, $\delta_2(x) = -z + \lambda y$. From [26, Example 7.2] we have that $B(\lambda)$ becomes a Hopf algebra via

$$\begin{aligned}\Delta(x) &= 1 \otimes x + x \otimes 1, & \varepsilon(x) &= 0, \\ \Delta(y) &= 1 \otimes y + y \otimes 1, & \varepsilon(y) &= 0, \\ \Delta(z) &= 1 \otimes z + x \otimes y + z \otimes 1, & \varepsilon(z) &= 0, \\ S(x) &= -x, & S(y) &= -y, & S(z) &= -z + xy.\end{aligned}$$

Note that $B(\lambda)$ is a connected Hopf algebra of GK- dimension three. (See [26, Proposition 7.3]).

Example 17. The Jordan plane $A = \mathbb{K}\langle x, y \rangle / \langle yx - xy - x^2 \rangle$ is a skew PBW extension of $\mathbb{K}[x]$. For more details and some properties of A see [5]. Note that the Jordan Plane is a coideal subalgebra of a connected Hopf algebra given in Example 16 (see [4, Example 5.5.9]). It is well known, the Jordan plane cannot be supplied with a comultiplication with respect to which it is a Hopf algebra. Therefore the Jordan plane is not a Hopf skew PBW extension.

Example 18. *Tensor algebra.* The tensor algebra $T(V)$ of a vector space V is a Hopf algebra with comultiplication $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\Delta(1) = 1 \otimes 1$, counit $\varepsilon(x) = 0$, antipode $S(x) = -x$, for all x in V (and extended to higher tensor powers). If $\dim(V) \geq 2$ then $T(V)$ is not commutative, $T(V)$ is cocommutative. Since a symmetric algebra and an exterior algebra are quotients of a tensor algebra, then these algebras are also Hopf algebras with this definition of the comultiplication, counit and antipode. Note that if $\dim \mathbb{K} > 1$, then $T(V)$ is not a skew PBW extension.

Example 19. Consider the polynomial algebra (according to [11])

$$A(X) = \mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}].$$

As a vector space, it's basis is

$$\{x_{11}^i x_{12}^j x_{21}^k x_{22}^l \mid i, j, k, l \in \mathbb{N}\}$$

The multiplication of two elements of $A(X)$ is the usual multiplication of polynomials, i.e., $\mu(x_{ij} \otimes x_{kl}) = x_{ij} x_{kl}$. If the comultiplication and counit are defined in $A(X)$ in the following manner,

$$\begin{aligned}\Delta(x_{ij}) &= x_{i1} \otimes x_{1j} + x_{i2} \otimes x_{2j} \\ \varepsilon(x_{ij}) &= \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}\end{aligned}$$

then we get a coalgebra.

We extend the action of Δ to rest of $A(X)$ by defining it to be an algebra morphism, i.e.,

$$\Delta(x_{ij} x_{kl}) = \Delta(x_{ij}) \Delta(x_{kl}).$$

Let

$$d = x_{11} x_{22} - x_{12} x_{21} \in A(X),$$

then

$$\begin{aligned}\Delta(d) &= \Delta(x_{11} x_{22} - x_{12} x_{21}) \\ &= \Delta(x_{11})(x_{22}) - \Delta(x_{12}) \Delta(x_{21}) \\ &= (x_{11} \otimes x_{11} + x_{12} \otimes x_{21})(x_{21} \otimes x_{12} + x_{22} \otimes x_{22}) \\ &\quad - (x_{11} \otimes x_{12} + x_{12} \otimes x_{22})(x_{21} \otimes x_{11} + x_{22} \otimes x_{21}) \\ &= (x_{11} \otimes x_{11})(x_{21} \otimes x_{12}) + (x_{11} \otimes x_{11})(x_{22} \otimes x_{22}) \\ &\quad + (x_{12} \otimes x_{21})(x_{21} \otimes x_{12}) + (x_{12} \otimes x_{21})(x_{22} \otimes x_{22}) \\ &\quad - (x_{11} \otimes x_{12})(x_{21} \otimes x_{11}) - (x_{11} \otimes x_{12})(x_{22} \otimes x_{21}) \\ &\quad - (x_{12} \otimes x_{22})(x_{21} \otimes x_{11}) - (x_{12} \otimes x_{22})(x_{22} \otimes x_{21}) \\ &= (x_{11} x_{21} \otimes x_{11} x_{12}) + (x_{11} x_{22} \otimes x_{11} x_{22}) \\ &\quad + (x_{12} x_{21} \otimes x_{21} x_{12}) + (x_{12} x_{22} \otimes x_{21} x_{22}) \\ &\quad - (x_{11} x_{21} \otimes x_{12} x_{11}) - (x_{11} x_{22} \otimes x_{12} x_{21}) \\ &\quad - (x_{12} x_{21} \otimes x_{22} x_{11}) - (x_{12} x_{22} \otimes x_{22} x_{21}) \\ &= (x_{11} x_{22} \otimes x_{11} x_{22}) - (x_{11} x_{22} \otimes x_{12} x_{21}) \\ &\quad + (x_{12} x_{21} \otimes x_{21} x_{12}) - (x_{12} x_{21} \otimes x_{22} x_{11}) \\ &= x_{11} x_{22} \otimes (x_{11} x_{22} - x_{12} x_{21}) \\ &\quad + x_{12} x_{21} \otimes (x_{21} x_{12} - x_{22} x_{11}) \\ &= x_{11} x_{22} \otimes (x_{11} x_{22} - x_{12} x_{21}) \\ &\quad - x_{12} x_{21} \otimes (x_{22} x_{11} - x_{21} x_{12}) \\ &= (x_{11} x_{22} - x_{12} x_{21}) \otimes (x_{11} x_{22} - x_{12} x_{21}) \\ &= d \otimes d,\end{aligned}$$

and $\varepsilon(d) = 1$.

Thus d is group-like. Note that $A(X)$ is a bialgebra but $A(X)$ has no antipode. So, we construct a new space

$$A(G) = \mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}, d^{-1}].$$

The comultiplication and counit are defined as in $A(X)$. Note that the relations

$$\begin{aligned}S(x_{11}) &= x_{22} d^{-1}, \\ S(x_{22}) &= x_{11} d^{-1}, \\ S(x_{12}) &= -x_{12} d^{-1}, \\ S(x_{21}) &= -x_{21} d^{-1},\end{aligned}$$

defines an antipode for $A(G)$ and makes $A(G)$ a Hopf algebra. Note that $A(X)$ is a skew PBW extension, but $A(G)$ is not a skew PBW extension.

2.4 Hopf pre-commutative skew PBW extensions

Pre-commutative skew PBW extensions were defined by Suárez in [20].

Definition 20 ([20], Definition 2.5-(a)). Let $A = \sigma(R)\langle x_1, \dots, x_n \rangle$ be a skew PBW extension of R . A is called *pre-commutative* if the condition (iv) in Definition 12 is replaced by the following:

For any $1 \leq i, j \leq n$, there exists $c_{ij} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{ij} x_i x_j \in Rx_1 + \dots + Rx_n. \quad (9)$$

Theorem 21. Let $A = \sigma(\mathbb{K})\langle x_1, \dots, x_n \rangle$ be a precommutative skew PBW extension of \mathbb{K} , with $c_{ij} = 1$. Then A is a cocommutative Hopf algebra.

Proof. Suppose that $A = \sigma(\mathbb{K})\langle x_1, \dots, x_n \rangle$ be a precommutative skew PBW extension of \mathbb{K} , with $c_{ij} = 1$ for $1 \leq i, j \leq n$. Then $x_j r - r x_i = 0$ and $x_i x_j - x_j x_i \in \mathbb{K}x_1 + \dots + \mathbb{K}x_n$, for any $r \in \mathbb{K}$ and $1 \leq i, j \leq n$. A is a Hopf algebra with comultiplication $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$ for $1 \leq i \leq n$, this rule can be uniquely extended to all A . The counit ε is given by $\varepsilon(x_i) = 0$, for $1 \leq i \leq n$ and extended to A . The antipode S is given by $S(x_i) = -x_i$. Let τ the flip map, then $\tau\Delta(x_i) = \tau(x_i \otimes 1 + 1 \otimes x_i) = 1 \otimes x_i + x_i \otimes 1 = \Delta(x_i)$ for $1 \leq i \leq n$. Then A is cocommutative.

Example 22. The universal enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra \mathfrak{g} (see Example 14) is a pre-commutative skew PBW extension of \mathbb{K} .

Example 23. q -Heisenberg algebra. The algebra $H_n(q)$ (with $q \in \mathbb{K} \setminus \{0\}$) is generated by the set of variables $x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n$ subject to the relations:

$$\begin{aligned} x_j x_i &= x_i x_j, & z_j z_i &= z_i z_j, & y_j y_i &= y_i y_j, & 1 \leq i, j \leq n; \\ z_j y_i &= y_i z_j, & z_j x_i &= x_i z_j, & y_j x_i &= x_i y_j, & i \neq j; \\ z_j y_i &= q y_i z_j, & z_j x_i &= q^{-1} x_i z_j + y_j, & 1 \leq i \leq n, \\ y_j x_i &= q x_i y_j, & 1 \leq i \leq n, \end{aligned}$$

Then $H_n(q) \cong \sigma(\mathbb{K}[y_1, \dots, y_n])\langle x_1, \dots, x_n; z_1, \dots, z_n \rangle$. Note that $H_n(q)$ is isomorphic to the iterated Ore extension

$\mathbb{K}[x_1, \dots, x_n][y_1; \sigma_1] \cdots [y_n; \sigma_n][z_1; \theta_1, \delta_1] \cdots [z_n; \theta_n, \delta_n]$ on the commutative polynomial ring $\mathbb{K}[x_1, \dots, x_n]$:

$$\begin{aligned} \theta_j(z_i) &:= z_i, & \delta_j(z_i) &:= 0, & 1 \leq i < j \leq n, \\ \theta_j(y_i) &:= y_i, & \delta_j(y_i) &:= 0, & i \neq j, \\ \theta_j(x_i) &:= x_i, & \delta_j(x_i) &:= 0, & i \neq j, \\ \theta_i(y_i) &:= q y_i, & \delta_i(y_i) &:= 0, & 1 \leq i \leq n, \\ \theta_i(x_i) &:= q^{-1} x_i, & \delta_i(x_i) &:= y_i, & 1 \leq i \leq n, \\ \sigma_j(y_i) &:= y_i, & 1 \leq i < j \leq n, \\ \sigma_j(x_i) &:= x_i, & i \neq j, \\ \sigma_i(x_i) &:= q x_i, & 1 \leq i \leq n. \end{aligned}$$

Since $\delta_i(x_i) = y_i \notin \mathbb{K}[x_1, \dots, x_n]$, then $H_n(q)$ is not a skew PBW extension of $\mathbb{K}[x_1, \dots, x_n]$, however, with respect to \mathbb{K} , $H_n(q)$ satisfies the conditions of Definition 12-(iii), and hence, $H_n(q)$ is a skew PBW extension of \mathbb{K} , i.e.,

$$H_n(q) = \sigma(\mathbb{K})\langle x_1, \dots, x_n; y_1, \dots, y_n; z_1, \dots, z_n \rangle.$$

Note that $H_n(q)$ is a pre-commutative skew PBW extension (see [23, Page 187]).

More examples of pre-commutative skew PBW extension of \mathbb{K} can be found in [20, 23].

Remark 24. If the hypothesis that A is a skew PBW extension of \mathbb{K} and that $c_{ij} = 1$ do not have, the Theorem 21 is false. Indeed:

1. The Jordan plane (Example 17) is a pre-commutative skew PBW extension (see [23, Page 187]) but this does not have a Hopf algebra structure. Moreover the Jordan plane is not a skew PBW extension of \mathbb{K} .
2. The quantum plane is the free algebra generated by x, y and the relation $yx = c_{1,2}xy$, with $c_{1,2} \in \mathbb{K} \setminus \{0\}$. This algebra is a pre-commutative skew PBW extension of \mathbb{K} . If $c_{1,2} \neq 1$ then the quantum plane cannot have a Hopf algebra structure (see [19, Theorem 4.5.2.]).

Example 25. Jordan plane. The Jordan plane A is the free algebra generated by x, y and relation $yx = xy + x^2$, so $A = \mathbb{K}\langle x, y \rangle / \langle yx - xy - x^2 \rangle \cong \sigma(\mathbb{K}[x])\langle y \rangle$.

3. HOPF ORE EXTENSION AND SKEW PBW EXTENSIONS

In this section we relate P-Hopf Ore extensions, Hopf Ore extensions and iterated Hopf Ore extensions, generalized Hopf Ore extensions and Hopf skew PBW extensions.

3.1 Panov Hopf Ore extensions

Panov in [10] defined the Hopf Ore extensions (here called P-Hopf Ore extensions) in order to study Ore extensions of Hopf algebras in the general setting.

Definition 26 ([10], Definition 1.0). Let R and $B = R[x; \sigma, \delta]$ be Hopf algebras. Then $B = R[x; \sigma, \delta]$ is called the *P-Hopf Ore extension* if $\Delta(x) = x \otimes r_1 + r_2 \otimes x$ for some $r_1, r_2 \in R$ and R is a Hopf subalgebra in B .

Remark 27. Let $B = R[x; \sigma, \delta]$ be a P-Hopf Ore extension with $\Delta(x) = x \otimes r_1 + r_2 \otimes x$ for some $r_1, r_2 \in R$.

- (i) r_1 and r_2 are group-like elements (see [10, Page 401]).
- (ii) Replacing the generating element x by $x' = xr_1^{-1}$, we see that $\Delta(x') = x' \otimes 1 + r \otimes x'$. Preserving the above notation, we assume in what follows that the element x in the P-Hopf Ore extension satisfies the relation $\Delta(x) = x \otimes 1 + r \otimes x$ for some group-like element $r \in R$. As usual, $Ad_r(a) = raS(r) = rar^{-1}$ (see [10, Page 402]).
- (iii) $\varepsilon(x) = 0$ and $S(x) = -r^{-1}x$, where $r^{-1} = S(r)$, for some group-like element $r \in R$ (see [10, Lemma 1.1]).
- (iv) If $\Delta(b) = b \otimes 1$, then $b \in \mathbb{K}$.

Proposition 28 ([10], Theorem 1.3). *The Hopf algebra $B = R[x; \sigma, \delta]$ is a P-Hopf Ore extension with $\Delta(x) = x \otimes 1 + r \otimes x$ if and only if*

- (i) *There is an algebra homomorphism (character) $\chi : R \rightarrow \mathbb{K}$ such that $\sigma(a) = \chi(a_1)a_2$, for any $a \in R$.*
- (ii) *The relations $\sigma(a) = \chi(a_1)a_2 = (rar^{-1})\chi(a_2)$ hold for all $a \in R$;*

- (iii) *The σ -derivation δ satisfies the relation $\Delta(\delta(a)) = \delta(a_1) \otimes a_2 + ra_1 \otimes \delta(a_2)$ for any $a \in R$.*

The condition (i) in Proposition 28 means that σ is a left winding automorphism τ'_χ of R , and the condition (ii) means that this left winding automorphism is also equal to right winding automorphism composed with the map of conjugation by r , i.e., $\tau'_\chi = Ad_r \circ \tau'_\chi$. Note that $\chi(a) := \sigma(a_1)S(a_2)$, where χ is as in Proposition 28 (see [10, Page 403]). If $B = R[x; \sigma, \delta]$ is a P-Hopf Ore extension and if R is a cocommutative Hopf algebra, then r belongs to the center of R (see [10, Corollary 1.4]).

Example 29. $\mathbb{K}[x]$ is a P-Hopf Ore extension and a Hopf skew PBW extension (see Example 13).

Remark 30. Note that the Hopf skew PBW extension $A(\lambda_1, \lambda_2, \alpha) = \mathbb{K}[x, y][\zeta; \sigma, \delta]$ (Example 15) is an Ore extension of a Hopf algebra $R = \mathbb{K}[x, y]$, but $A(\lambda_1, \lambda_2, \alpha)$ is not a P-Hopf Ore extension since $\Delta(\zeta) = 1 \otimes \zeta + x \otimes y + \zeta \otimes 1$.

3.2 Hopf Ore extensions

Brown, O'Hagan, Zhang and Zhuang in [2] studied when an Ore extension $B = R[x; \sigma, \delta]$ of a Hopf algebra R admits a Hopf algebra structure, generalizing the study made by Panov in [10]. In what follows, we shall always assume that the antipode is bijective.

Definition 31 ([2], Definition 2.1). Let R be a Hopf algebra. A *Hopf Ore Extension* of R is an algebra B such that

- (i) B is a Hopf algebra with Hopf subalgebra R ;
- (ii) There exist an algebra automorphism σ and a σ -derivation δ of R such that $B = R[x; \sigma, \delta]$;
- (iii) there are $a, b \in R$ and $v, w \in R \otimes R$ such that

$$\Delta(x) = a \otimes x + x \otimes b + v(x \otimes x) + w, \quad (10)$$

Remark 32. The commutative polynomial algebra $H = \mathbb{K}[x, y, \zeta] = \mathbb{K}[y, \zeta][x]$ (as in [2, Page 2411]) is Hopf Ore extension with coefficient Hopf algebra $R = \mathbb{K}[y, \zeta]$, but

$$\Delta(x) = x \otimes 1 + 1 \otimes x + y \otimes \zeta.$$

Then H is not a P-Hopf Ore extension. Note that H is a Hopf skew PBW extension.

Proposition 33. *Every P-Hopf Ore extension is a Hopf Ore extension.*

Proof. Let $H = R[x; \sigma, \delta]$ be P-Hopf Ore extension. Then R is a Hopf algebra, H is a Hopf algebra and $\Delta(x) = x \otimes 1 + r \otimes x$ for some group-like element $r \in R$. Thus, $\Delta(x) = x \otimes 1 + r \otimes x + v(x \otimes x) + w$, where $v = 0 = w$. Therefore H is a Hopf Ore extension.

Proposition 34 ([2], Proposition 2.8). *Let R be a connected Hopf algebra and let $B = R[x; \sigma, \delta]$ be a Hopf algebra containing R as a Hopf subalgebra. Then*

$$\Delta(x) = 1 \otimes x + x \otimes 1 + w$$

for some $w \in R \otimes R$. As a consequence, B is a Hopf Ore extension of R and is a connected Hopf algebra.

Example 35. The connected Hopf skew PBW extension $A(\lambda_1, \lambda_2, \alpha) = \sigma(\mathbb{K}[x, y])\langle \zeta \rangle = \mathbb{K}[x, y][\zeta; \sigma, \delta]$ of the Example 15 is a Hopf Ore extension of $R = \mathbb{K}[x, y]$, since $\Delta(\zeta) = 1 \otimes \zeta + \zeta \otimes 1 + w$, where $w = x \otimes y$.

3.3 Iterated Hopf Ore Extensions

Definition 36. An iterated Hopf Ore extension of \mathbb{K} is a Hopf algebra

$$H = \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_n; \sigma_n, \delta_n] \quad (11)$$

where

- (i) H is a Hopf algebra;
- (ii) $H_{(i)} := \mathbb{K}\langle x_1, \dots, x_i \rangle$ is a Hopf subalgebra of H for $i = 1, \dots, n$;
- (iii) σ_i is an algebra automorphism of $H_{(i-1)}$, and δ_i is a σ_i -derivation of $H_{(i-1)}$, for $i = 2, \dots, n$.

Example 37. Let \mathfrak{g} be a finite dimensional Lie algebra with basis $\{x_1, \dots, x_n\}$ and $U(\mathfrak{g})$ the Hopf skew PBW extension of \mathbb{K} as in Example 14. If \mathfrak{g} is a solvable Lie algebra then $U(\mathfrak{g})$ is an iterated Hopf Ore extension, with $\sigma_i = id$ for all i (see [2, Example 3.1-(ii)]).

Example 38. The Hopf skew PBW extension $B(\lambda) = \sigma(\mathbb{K}[y])\langle x, \zeta \rangle = \mathbb{K}[y][x; \sigma_1, \delta_1][\zeta; \sigma_2, \delta_2]$ as in Example 16 is an iterated Hopf Ore extension.

Example 39. Let D be the algebra generated by x, y, ζ, w satisfying the following relations,

$$\begin{aligned} yx - xy &= \zeta x - x\zeta = \zeta y - y\zeta = 0, \\ wx - xw &= a_{11}x + a_{12}y, \\ wy - yw &= a_{21}x + a_{22}y, \\ w\zeta - \zeta w &= (a_{11} + a_{22})\zeta + \xi_1x + \xi_2y, \end{aligned}$$

where $a_{ij}, \xi_i \in \mathbb{K}$. Then D becomes a bialgebra via

$$\varepsilon(x) = \varepsilon(y) = \varepsilon(\zeta) = \varepsilon(w) = 0$$

$$\begin{aligned} \Delta(x) &= 1 \otimes x + x \otimes 1, \\ \Delta(y) &= 1 \otimes y + y \otimes 1, \\ \Delta(\zeta) &= 1 \otimes \zeta + x \otimes y - y \otimes x + \zeta \otimes 1, \\ \Delta(w) &= 1 \otimes w + w \otimes 1 \\ &\quad + \theta_1(\zeta \otimes x - x \otimes \zeta + x \otimes xy + xy \otimes x) \\ &\quad + \theta_2(y \otimes \zeta - \zeta \otimes y + xy \otimes y + y \otimes xy), \end{aligned}$$

where $\theta_i \in \mathbb{K}$ and at least one of them is non-zero (see [25, Example 4.4]). Moreover D is an iterated Ore extension $\mathbb{K}[x][y; \delta_1][\zeta; \delta_2][w; \sigma_3, \delta_3]$ (see [25, Proposition 4.8-(a)]). Therefore D is a Hopf skew PBW extension. Note that the Lie algebra $P(D)$ is $P(D) = \mathbb{K}x + \mathbb{K}y$ (see [25, Proposition 4.8-(e)]).

We denote the Gelfand-Kirillov dimension of H by $GK \dim(H)$.

Theorem 40. *Let H be an iterated Hopf Ore extension with $1 \leq GK \dim(H) \leq 4$. Then H is a Hopf skew PBW extension.*

Proof. Let H be an iterated Hopf Ore extension.

- (i) If $GK \dim(H) = 1$ then $H = \mathbb{K}[x]$ (see [2, Theorem 3.3-(ii)]). Note that $\mathbb{K}[x]$ is a Hopf skew PBW extension (see Example 13).
- (ii) If $GK \dim(H) = 2$ then $H = U(\mathfrak{g})$, where \mathfrak{g} is one of the two Lie algebras of dimension 2 (see [2, Theorem 3.3-(iii)]). $U(\mathfrak{g})$ is a Hopf skew PBW extension (see Example 14).
- (iii) Suppose that $GK \dim(H) = 3$. Then by [2, Theorem 3.3-(iv)] H is isomorphic as a Hopf algebra to one (and only one) of the following:

- (a) the enveloping algebra of a three-dimensional Lie algebra;
- (b) the algebras $A(0, 0, 0), A(0, 0, 1), A(1, 1, 1), A(1, \lambda, 0), \lambda \in \mathbb{K}$;
- (c) the algebras $B(\lambda), \lambda \in \mathbb{K}$.

Note that the enveloping algebra in item (a) is a Hopf skew PBW extension (see Example 14). The algebras of item (b) are particular cases of the family of algebras of the Example 15, which are Hopf skew PBW extension. Now, $B(\lambda)$ is also a Hopf skew PBW extension (see Example 16).

- (iv) If $GK \dim(H) = 4$ then by [25, Corollary 4.25] H is isomorphic to $U(\mathfrak{g})$ (as an algebra), for some Lie algebra \mathfrak{g} . For the Example 14 we have that H is a Hopf skew PBW extension.

Remark 41. Note that not all skew PBW extension of \mathbb{K} is an iterated Hopf Ore extension. Indeed: If \mathfrak{g} is a semisimple Lie algebra with a simple factor not isomorphic to $\mathfrak{sl}(2, \mathbb{K})$, then $U(\mathfrak{g})$ is *not* an iterated Hopf Ore extension (see [2, Example 3.1-iv]). Note that $U(\mathfrak{g})$ is a skew PBW extension. The infinite families of Hopf skew PBW extensions in Examples 15 and 16 are iterated Hopf Ore extensions which are neither commutative nor cocommutative.

3.4 Generalized Hopf Ore extensions

Definition 42 ([24], Definition 1.1). Let R be a Hopf algebra and $H = R[x; \sigma, \delta]$ an Ore extension of R . If there is a Hopf algebra structure on H such that R is a Hopf subalgebra of H and

$$\Delta(x) = x \otimes r_1 + \zeta \otimes y + r_2 \otimes x$$

for some $r_1, r_2, \zeta, y \in R$, then H is called a *generalized Hopf Ore extension* of R . In this case, we also say that H has a Hopf algebra structure determined by (r_1, r_2, ζ, y) .

Example 43. Let A be the algebra generated by elements x, y satisfying the relation $yx - xy = -x$. $A = \mathbb{K}[x][y; \sigma, \delta]$ is an Ore extension of $\mathbb{K}[x]$ with $\sigma(x) = x$ and $\delta(x) = -x$. The polynomial algebra $\mathbb{K}[x]$ has a Hopf algebra structure with x primitive (see Example 13), i.e., $\Delta(x) = 1 \otimes x + x \otimes 1$. Therefore, A is a generalized Hopf Ore extension.

Proposition 44. *The Hopf skew PBW extensions $A(\lambda_1, \lambda_2, \alpha)$ and $B(\lambda)$ of the Examples 15 and 16 are generalized Hopf Ore extensions of the enveloping algebras of some 2-dimensional Lie algebras.*

Proof. This is a consequence of [24, Proposition 2.16] and [24, Remark 2.17].

Theorem 45. *P-Hopf Ore extension \Rightarrow generalized Hopf Ore extension \Rightarrow Hopf Ore extension \Rightarrow Hopf skew PBW extension.*

Proof. Let $H = R[x; \sigma, \delta]$ be P-Hopf Ore extension. Then R is a Hopf algebra, H is a Hopf algebra and $\Delta(x) = x \otimes r_1 + r_2 \otimes x$ for some group-like element $r_1, r_2 \in R$. Thus, $\Delta(x) = x \otimes r_1 + r_2 \otimes x + \zeta \otimes y$, with $\zeta \otimes y = 0$. Therefore H is a generalized Hopf Ore extension. Now, if $H = R[x; \sigma, \delta]$ is a generalized Hopf Ore extension, there is a Hopf algebra structure on H such that R is a Hopf subalgebra of H and $\Delta(x) = x \otimes r_1 + \zeta \otimes y + r_2 \otimes x$, for some $r_1, r_2, \zeta, y \in R$. Thus for $v = 0$, $\Delta(x) = x \otimes r_1 + r_2 \otimes x + v(x \otimes x) + w$, where $w = \zeta \otimes y$. Therefore H is a Hopf Ore extension. Since Ore extensions are skew PBW extensions and $H = R[x; \sigma, \delta]$ is a Hopf algebra, then H is a Hopf skew PBW extension.

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